COMMUTING ROW CONTRACTIONS WITH POLYNOMIAL CHARACTERISTIC FUNCTIONS

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To the memory of Ciprian Foias

Abstract. A characteristic function is a special operator-valued analytic function defined on the open unit ball of $\mathbb{C}^n$ associated with an n-tuple of commuting row contraction on some Hilbert space. In this paper, we continue our study of the representations of n-tuples of commuting row contractions on Hilbert spaces, which have polynomial characteristic functions. Gleason's problem plays an important role in the representations of row contractions. We further complement the representations of our row contractions by proving theorems concerning factorizations of characteristic functions. We also emphasize the importance and the role of the noncommutative operator theory and noncommutative varieties to the classification problem of polynomial characteristic functions.

1. Introduction

Identifying and then computing a complete unitary invariant of (tuples of) bounded linear operators on Hilbert spaces is one of the central objects in operator theory. From this point of view, the notion of characteristic function of contractions on Hilbert spaces stands out in its breadth of applications in function theory and operator theory.

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators on a Hilbert space $\mathcal{H}$, and let T be a row contraction (that is, $\sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}$). The characteristic function of T is the $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$-valued analytic function

$$\theta_T(z_1, \ldots, z_n) = [-T + D_{T^*}\big(I_{\mathcal{H}} - \sum_{i=1}^n z_i T_i^*\big)^{-1} Z D_T]|_{\mathcal{D}_T},$$

for all $(z_1, \ldots, z_n) \in \mathbb{B}^n$, where $\mathbb{B}^n$ denotes the open unit ball in $\mathbb{C}^n$, $Z = (z_1 I_{\mathcal{H}}, \ldots, z_n I_{\mathcal{H}})$ is a row operator, $D_T = (I - T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{\text{ran}} D_T$ (see Section 2 for more details).

In particular, if $n = 1$, then the above definition of θ_T becomes the well known and classical Sz.-Nagy and Foias characteristic function of the single contraction T [19]. In this case, clearly, θ_T admits a power series expansion on the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. This, of course, immediately raises the natural question of the relationship between the class of polynomial characteristic functions and the structure of corresponding contractions. To some extent, the work of Foias and the third author [7] gives a satisfactory answer to this question. For instance: The characteristic function θ_T of a completely nonunitary contraction T on a

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separable, infinite dimensional, complex Hilbert space $\mathcal{H}$ is a polynomial if and only if there exist three closed subspaces $\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_{-1}$ of $\mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$, a pure isometry S on $\mathcal{H}_1$, a nilpotent N on $\mathcal{H}_0$, and a pure co-isometry C on $\mathcal{H}_{-1}$, such that T admits the following matrix representation

$$T = \begin{bmatrix} S & * & * \\ 0 & N & * \\ 0 & 0 & C \end{bmatrix}.$$

Moreover, the dimension of $\ker S^*$ and dimension of $\ker C$ are unitary invariants of T and that N, up to a quasi-similarity, is uniquely determined by T (see [7, Sections 4 and 5]). In the follow-up paper, Foias, Pearcy and the third author [8] proved the following analytic result: If θ_T is a polynomial of degree m, then there exist a Hilbert space $\mathcal{M}$, a nilpotent operator N of order m, a coisometry $V_1 \in \mathcal{B}(\mathcal{D}_{N^*} \oplus \mathcal{M}, \mathcal{D}_{T^*})$, and an isometry $V_2 \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_N \oplus \mathcal{M})$, such that

$$\theta_T = V_1 \begin{bmatrix} \theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2.$$

On the other hand, the approach of [7] was continued and extended to n-tuples of noncommuting row contractions setting by Popescu in [17]. Also, the results of [8] were further extended to Popescu's noncommutative setting in [9].

It is worthwhile to note that Popescu (see [16] and other references therein) first recognized that the notion of characteristic functions, a special class of multi-analytic operators [14], plays a central role in multivariable operator theory and noncommutative function theory. Moreover, his approach to noncommutative varieties links up with the noncommutative operator theory and commutative operator theory (see [13, 16] and Section 5).

This paper aims to complete the classification problem of contractions, which admits polynomial characteristic functions. More precisely, we aim to classify n-tuples of commuting row contractions, which admits polynomial characteristic functions.

The question of the structure of n-tuples of commuting row contractions is important in its own right. However, on the other hand, Popescu's approach to noncommutative varieties unifies many analytic and geometric questions concerning n-tuples of commuting row contractions. From this point of view, it is also necessary to examine the noncommutative operator theoretic technique and the classifications of noncommuting row contractions admitting polynomial characteristic functions to our classification problem of tuples of commuting row contractions. As we will see, some of the present techniques and results are similar to the one variable case and the noncommutative case. However, commutativity property (a constrained property, as identified by Popescu in [13, 16]) brings out more intrinsic function theoretic features to the classification problem. Indeed, natural and satisfactory versions of the classification problem (for instance, see Theorem 3.5) are related to the notion of Gleason's problem (see Definition 3.4).

More specifically, suppose $T = (T_1, \ldots, T_n)$ be a commuting row contraction on a Hilbert space $\mathcal{H}$. Theorem 3.3 records the following general observation: Suppose the characteristic function of T is a polynomial of degree m. If

$$\mathcal{M} = \overline{\text{span}}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, |\alpha| \geq m, \alpha \in \mathbb{Z}_+^n\},$$

and $M_i := T_i|_{\mathcal{M}}$ for all $i = 1, \ldots, n$, and

$$\mathcal{N} = \overline{\text{span}}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, |\alpha| = m, \alpha \in \mathbb{Z}_+^n\},$$

then $\mathcal{M}$ is a joint closed invariant subspace for T and the restriction tuple $M = (M_1, \ldots, M_n)$ is a commuting pure partial isometry on $\mathcal{M}$ (see the end of Section 2 for the definition of pure tuples). Moreover

$$\mathcal{M} = \overline{\text{span}}\{M^\alpha \mathcal{N} : \alpha \in \mathbb{Z}_+^n\},$$

and

$$\mathcal{N} = \mathcal{M} \ominus \big(\sum_{i=1}^n M_i \mathcal{M}\big),$$

and $\mathcal{M}$ is the minimal closed joint M-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$.

Comparing this construction with that of noncommuting tuples of operators [17] (including the $n = 1$ case [7]), one may be tempted to conclude that the n-tuple M on $\mathcal{M}$, up to unitary equivalence, is simply the multiplication tuple $(M_{z_1}, \ldots, M_{z_n})$ on $H_n^2(\mathcal{N})$, the $\mathcal{N}$-valued Drury-Arveson shift (see Section 2). However, for n-tuples of commuting row contractions, $n > 1$, this is not true in general. This problem is connected to Gleason's property of functions on the unit ball [5, Section 2]. This motivates us to isolate the specific class of operators, which we call regular tuples operators (see Definition 3.4): T is regular if there exists $\epsilon > 0$ such that for any $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with $\|\boldsymbol{z}\|_{\mathbb{C}^n} < \epsilon$, the subspace

$$(T - Z)\mathcal{H}^n := \sum_{i=1}^n (T_i - z_i I_{\mathcal{H}})\mathcal{H},$$

is closed in $\mathcal{H}$ and

$$\mathcal{H} = (T - Z)\mathcal{H}^n \dot{+} \Big(\mathcal{H} \ominus \sum_{i=1}^n T_i \mathcal{H}\Big).$$

By keeping in mind the regular tuples of operators (corresponding to Gleason's property of functions), in Theorem 3.6, we present the following complete picture: Suppose the characteristic function of a commuting row contraction T on $\mathcal{H}$ is a polynomial of degree m. With $\mathcal{M}$ as above, we set

$$\mathcal{H}_{nil} = \overline{\text{span}}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, \alpha \in \mathbb{Z}_+^n\} \ominus \mathcal{M},$$

and

$$\mathcal{H}_c = \{h \in \mathcal{H} : \sum_{|\alpha| = k} \|T^{*\alpha} h\|^2 = \|h\|^2 \text{ for all } k \in \mathbb{Z}_+\}.$$

Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{H}_{\text{nil}} \oplus \mathcal{H}_c$ and T_i admits the following matrix decomposition

$$T_i = \begin{bmatrix} M_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}, \tag{1.1}$$

for all $i = 1, \ldots, n$, where $M = (M_1, \ldots, M_n)$ on $\mathcal{M}$ is a pure row contraction, N on $\mathcal{H}_{\rm nil}$ is a commuting nilpotent tuple of order less than or equal to m and W on $\mathcal{H}_c$ is a commuting spherical co-isometry. Moreover,

$$\sum_{i=1}^{n} M_i M_i^* = I_{\mathcal{M}} - P_{\mathcal{N}},$$

where $\mathcal{N} = \mathcal{M} \ominus \Big(\sum_{i=1}^{n} T_i \mathcal{M}\Big)$. If, in addition, M is regular, then it is a Drury-Arveson shift.

We stress that our results for commutating tuples of operators does not follow from Popescu's noncommutative theory (see Remark 3.8). The above set of results is the main content of Section 3.

Section 4 is devoted to the study of factorizations of characteristic functions of n-tuples of noncommutative row contractions. Given a noncommuting row contraction $T = (T_1, \ldots, T_n)$, we denote by Θ_T the characteristic function of T. Motivated by [8, Theorem 1.3], in Theorem 4.3, we prove the following factorization result: Let $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ be Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$. Assume that $T = (T_1, \ldots, T_n)$ is a row contraction on $\mathcal{H}$ and

$$T_i = \begin{bmatrix} S_i & * & * \\ 0 & N_i & * \\ 0 & 0 & C_i \end{bmatrix},$$

for all $i = 1, \ldots, n$. Then S, N and C are n-tuples of row contractions on $\mathcal{H}_1, \mathcal{H}_0$ and $\mathcal{H}_{-1}$, respectively, and there exist Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}$, and unitary operators

$$\tau_1 \in \mathcal{B}(\mathcal{D}_{N^*} \oplus \mathcal{E}, \mathcal{D}_C \oplus \mathcal{E}_1) \quad \text{and} \quad \tau_2 \in B(\mathcal{D}_{S^*} \oplus \mathcal{E}_2, \mathcal{D}_N \oplus \mathcal{E}),$$

such that Θ_T coincides with

$$\begin{bmatrix} \Theta_C & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}_1} \end{bmatrix} (I_\Gamma \otimes \tau_1) \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}} \end{bmatrix} (I_\Gamma \otimes \tau_2) \begin{bmatrix} \Theta_S & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}_2} \end{bmatrix}.$$

Finally, in view of Popescu [17, Theorem 1.1], in Corollary 4.5, we prove the following factorization result: Let T be a noncommutative row contraction on $\mathcal{H}$ such that the characteristic function Θ_T is a noncommutative polynomial of degree m. Then there exist a Hilbert space $\mathcal{E}$, a nilpotent row contraction $N = (N_1, \ldots, N_n)$ of order $\leq m$, such that

$$\Theta_T = G_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{\mathcal{E}}} \end{bmatrix} G_2,$$

where G_1 and G_2 are co-isometry and isometry in $\mathcal{B}(\Gamma \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), \Gamma \otimes \mathcal{D}_{T^*})$ and $\mathcal{B}(\Gamma \otimes \mathcal{D}_T, \Gamma \otimes (\mathcal{D}_N \oplus \mathcal{E}))$, respectively. The $n = 1$ case of this result comes from [8].

In Section 5, we continue our discussion of factorizations of characteristic functions in the setting of noncommutative varieties [13]. Here, we prove analogous results as developed in the early part of this paper in the setting of noncommutative varieties. It is well known that the class of commuting row contractions on Hilbert spaces can be realized through a particular noncommutative variety in the sense of Popescu [13]. From this point of view, in Theorem

5.4, we prove the following result: Let $T = (T_1, \dots, T_n)$ be an n-tuple of commuting row contraction on $\mathcal{H}$ such that θ_T is a polynomial of degree m, and let

$$\mathcal{M} = \overline{\text{Span}}\{T^\alpha D_{T^*}h : h \in \mathcal{H}, |\alpha| \geq m, \alpha \in \mathbb{Z}_+^n\}.$$

If T is regular, then there exist a Hilbert space $\mathcal{E}$, a co-isometry $G_1 \in \mathcal{B}(H_n^2 \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), H_n^2 \otimes \mathcal{D}_{T^*})$ and a partial isometry $G_2 \in \mathcal{B}(H_n^2 \otimes \mathcal{D}_T, H_n^2 \otimes (\mathcal{D}_N \oplus \mathcal{E}))$ such that

$$\theta_T = G_1 \begin{bmatrix} \theta_N & 0 \\ 0 & I_{H_n^2 \otimes \mathcal{D}_\mathcal{E}} \end{bmatrix} G_2.$$

Note that if $n = 1$, then the partial isometry G_2 becomes an isometry [8, Theorem 2.2].

In Section 6, we discuss some unitary invariants of n-tuples of commuting row contractions, which admit polynomial characteristic functions. The results here are motivated by the earlier results of Popescu [17]. The final section is devoted to an example to justify the regularity assumption on commuting tuples of row contractions, where the following introductory section, Section 2, briefly outlines a few key facts of Drury-Arveson space, n-tuples of commuting row contractions, and characteristic functions of commuting row contractions.

Finally, we remark that from the multivariable operator theory point of view, this is a sequel to the papers [7] and [8] by Foias, and Foias and Pearcy, respectively, and the third author.

2. Preliminaries

In this section, we recall basic definitions and notations used in the rest of the paper. Throughout the paper, Hilbert spaces will be denoted by $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{E}$, $\mathcal{E}_*$, etc. The set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ will be denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. When $\mathcal{H}_1 = \mathcal{H}_2$, one writes simply $\mathcal{B}(\mathcal{H}_1)$ instead of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$. Now let $\{T_1, \dots, T_n\} \subseteq \mathcal{B}(\mathcal{H})$. We say that $T = (T_1, \dots, T_n)$ is a *row contraction* (or *spherical contraction*) if the row operator $T : \mathcal{H}^n \to \mathcal{H}$, defined by

$$T(h_1, \dots, h_n) = \sum_{i=1}^n T_i h_i \qquad (h_1, \dots, h_n \in \mathcal{H}),$$

is a contraction. It is clear that T is a row contraction if and only if $\sum\limits_{i=1}^n \|T_i^* h\|^2 \leq \|h\|^2$ for all $h \in \mathcal{H}$, or equivalently $\sum_{i=1}^n T_i T_i^* \leq I_\mathcal{H}$. A row contraction T is said to be *commuting row contraction* if $T_i T_j = T_j T_i$ for $i, j = 1, \dots, n$.

A typical example of a commuting row contraction is the n-tuple of multiplication operator $(M_{z_1}, \dots, M_{z_n})$ on the *Drury-Arveson space* H_n^2, where H_n^2 is the reproducing kernel Hilbert space corresponding to the kernel

$$k(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Here $\mathbb{B}^n$ denotes the open unit ball in $\mathbb{C}^n$ and $\boldsymbol{z}$ (and $\boldsymbol{w}$ etc.) denotes an element in $\mathbb{C}^n$, that is, $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then

$$H^2_n = \{f = \sum_{\alpha \in \mathbb{Z}^n_+} a_\alpha \boldsymbol{z}^\alpha : a_\alpha \in \mathbb{C} \text{ and } \|f\|^2 := \sum_{\alpha \in \mathbb{Z}^n_+} \frac{|a_\alpha|^2}{\gamma_\alpha} < \infty\},$$

where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ and

$$\gamma_\alpha := \frac{|\alpha|!}{\alpha!} = \frac{(\sum_{i=1}^n \alpha_i)!}{\alpha_1! \cdots \alpha_n!},$$

is the multinomial coefficient. The $\mathcal{E}$-valued Drury-Arveson space will be denoted by $H^2_n(\mathcal{E})$. In this case, the representation of $H^2_n(\mathcal{E})$ is the same as H^2_n above but replacing $a_\alpha \in \mathbb{C}$ with $a_\alpha \in \mathcal{E}$ and $|a_\alpha|$ with $\|a_\alpha\|_{\mathcal{E}}$. Now identifying $H^2_n(\mathcal{E})$ with the Hilbert space tensor product $H^2_n \otimes \mathcal{E}$ (via $\boldsymbol{z}^\alpha \eta \mapsto \boldsymbol{z}^\alpha \otimes \eta$, $\alpha \in \mathbb{Z}^n_+$ and $\eta \in \mathcal{E}$), we see that $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_n(\mathcal{E})$ and $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ on $H^2_n \otimes \mathcal{E}$ are unitarily equivalent. We shall frequently make use of this identification. Given a commuting tuple $M = (M_1, \ldots, M_n)$ on a Hilbert space $\mathcal{H}$, we often say that M is a *Drury-Arveson shift* if there exists a Hilbert space $\mathcal{W}$ such that M and $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_n(\mathcal{W})$ are unitarily equivalent.

Also recall that a holomorphic function $\varphi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ is said to be a (Drury-Arveson) *multiplier* if

$$\varphi H^2_n(\mathcal{E}) \subseteq H^2_n(\mathcal{E}_*).$$

In this case, by virtue of the closed graph theorem, it follows that the multiplication operator $M_\varphi : H^2_n(\mathcal{E}) \to H^2_n(\mathcal{E}_*)$ (where $M_\varphi f = \varphi f$ for all $f \in H^2_n(\mathcal{E})$) is a bounded linear operator. The set of all multipliers will be denoted by $\mathcal{M}(\mathcal{E}, \mathcal{E}_*)$. It also follows that $\mathcal{M}(\mathcal{E}, \mathcal{E}_*)$ is a Banach space relative to the operator norm

$$\|\varphi\| := \|M_\varphi\|_{\mathcal{B}(H^2_n(\mathcal{E}), H^2_n(\mathcal{E}_*))} \qquad (\varphi \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)).$$

Now let $T = (T_1, \ldots, T_n)$ be a row contraction on $\mathcal{H}$. The *defect operators* and *defect spaces* of T are given by

$$D_T = (I - T^*T)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H}^n) \quad \text{and} \quad D_{T^*} = (I - TT^*)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H}),$$

and

$$\mathcal{D}_T = \overline{\text{ran}} D_T \subseteq \mathcal{H}^n \quad \text{and} \quad \mathcal{D}_{T^*} = \overline{\text{ran}} D_{T^*} \subseteq \mathcal{H},$$

respectively. For any commuting row contraction $T = (T_1, \ldots, T_n)$ on $\mathcal{H}$, the *characteristic function* of T is a $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$-valued analytic function $\theta_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ defined by

$$\theta_T(\boldsymbol{z}) = \Big(- T + D_{T^*} \big(I_{\mathcal{H}} - ZT^*\big)^{-1} Z D_T \Big)|_{\mathcal{D}_T} \qquad (\boldsymbol{z} \in \mathbb{B}^n), \tag{2.1}$$

where $Z = (z_1 I_{\mathcal{H}}, \ldots, z_n I_{\mathcal{H}})$ is a row operator on $\mathcal{H}$ and so $ZT^* = \sum\limits_{i=1}^n z_i T_i^*$ for all $\boldsymbol{z} \in \mathbb{B}^n$. Also we define $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ and $T^{*\alpha} = T_1^{*\alpha_1} \cdots T_n^{*\alpha_n}$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, and $P_j : \mathcal{H}^n \to \mathcal{H}$ by

$$P_j(h_1, \ldots, h_n) = h_j \qquad (h_1, \ldots, h_n \in \mathcal{H}).$$

Then

$$\theta_T(\boldsymbol{z}) = \Big(- T + D_{T^*} \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ j=1}}^{n} \gamma_\alpha T^{*\alpha} \boldsymbol{z}^{\alpha+e_j} P_j D_T \Big)|_{\mathcal{D}_T}.$$

If we define $\theta_{T,\alpha}$, the coefficient of $\boldsymbol{z}^\alpha$, $\alpha \in \mathbb{Z}_+^n$, in the Taylor series expansion of θ_T as

$$\theta_{T,\alpha} = \begin{cases} -T|_{\mathcal{D}_T} & \text{if } \alpha = 0 \\ \displaystyle\sum_{j=1}^{n} \gamma_{\alpha-e_j} D_{T^*} T^{*(\alpha-e_j)} P_j D_T|_{\mathcal{D}_T} & \text{if } \alpha \neq 0, \end{cases}$$

then $\theta_T(\boldsymbol{z}) = \sum_{|\alpha|\geq 0} \theta_{T,\alpha} \boldsymbol{z}^\alpha$, $\boldsymbol{z} \in \mathbb{B}^n$. In what follows, we adopt the standard convention that

$$\gamma_{\alpha-e_j} = 0 \quad \text{and} \quad T^{*(\alpha-e_j)} = I \qquad (\alpha \in \mathbb{Z}_+^n, \alpha_j = 0).$$

It is now natural to define polynomial characteristic functions. Let T be an n-tuple of commuting row contraction on $\mathcal{H}$ and let m be a natural number. We say that the characteristic function θ_T is a *polynomial of degree* m if

$$\theta_{T,\alpha} \neq 0,$$

for some $|\alpha| = m$ and $\theta_{T,\beta} = 0$ for all $|\beta| > m$. If $\theta_T(\boldsymbol{z}) \equiv -T|_{\mathcal{D}_T}$, $\boldsymbol{z} \in \mathbb{B}^n$, then we say that θ_T is a *polynomial of degree zero*. Throughout this paper, we make the convention that the degree of the *zero polynomial* is zero.

A commuting tuple $N = (N_1, \dots, N_n)$ on $\mathcal{H}$ is said to be *nilpotent of order* $m(> 1)$ if

$$N^\alpha = 0 \quad \text{and} \quad N^\beta \neq 0,$$

for all α in $\mathbb{Z}_+^n$ with $|\alpha| = m$ and for some β in $\mathbb{Z}_+^n$ such that $|\alpha| - |\beta| = 1$. For a commuting row contraction $T = (T_1, \dots, T_n)$ on $\mathcal{H}$ we define

$$\mathcal{H}_c := \Big\{ h \in \mathcal{H} : \sum_{|\alpha|=k} \|T^{*\alpha} h\|^2 = \|h\|^2 \quad \text{for all} \quad k \in \mathbb{Z}_+ \Big\}. \tag{2.2}$$

Clearly, $\mathcal{H}_c$ is a closed and joint $(T_1^*, \dots, T_n^*)$-invariant subspace. Moreover, $\mathcal{H}_c$ is *maximal*, that is, $\mathcal{H}_c$ is the largest closed subspace of $\mathcal{H}$ on which $T^* : \mathcal{H} \to \mathcal{H}^n$ acts isometrically. The row contraction T is said to be a *completely non-coisometric* (c.n.c) row contraction if $\mathcal{H}_c = \{0\}$. The row contraction T is said to be *pure* if

$$\lim_{k\to\infty} \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha|=k}} \|T^{*\alpha} h\|^2 = 0 \qquad (h \in \mathcal{H}).$$

As an example, we note that the multiplication tuple $(M_{z_1}, \dots, M_{z_n})$ on a vector-valued Drury-Arveson space $H_n^2(\mathcal{E})$ is a pure row contraction.

Finally, we recall that a pair of commuting n-tuples of row contractions $(T_1, \dots, T_n)$ and $(T_1^{'}, \dots, T_n^{'})$ are said to be *unitary equivalent* if there exists a unitary $U : \mathcal{H} \to \mathcal{H}^{'}$ such that $T_i = U T_i^{'} U^*$ for all $i = 1, \dots, n$.

3. Polynomial Characteristic Functions

This section presents the representations of n-tuples of commuting row contractions, which admits polynomial characteristic functions. Gleason's problem plays a crucial role in our consideration. We begin with the following key lemma.

Lemma 3.1. *Let $T = (T_1, \ldots, T_n)$ be a commuting row contraction on a Hilbert space $\mathcal{H}$. Suppose θ_T is a polynomial of degree m. If $\alpha \in \mathbb{Z}_+^n$ and $|\alpha| \geq m+1$, then*

$$T_i^*(T^\alpha D_{T^*}) = \frac{\alpha_i}{|\alpha|}(T^{\alpha - e_i} D_{T^*}),$$

for all $i \in \{1, \ldots, n\}$.

Proof. Fix $i \in \{1, \ldots, n\}$. For each $|\alpha| \geq m+1$, since $\theta_{T,\alpha}^* = 0$, it follows that

$$D_T^2 \sum_{j=1}^n \gamma_{\alpha - e_j} P_j^* T^{\alpha - e_j} D_{T^*} = 0.$$

Note that $P_j^* : \mathcal{H} \to \mathcal{H}^n$ is given by

$$P_j^*(h) = (0, \ldots, 0, \underbrace{h}_{\text{j-th position}}, 0, \ldots, 0),$$

for all $h \in \mathcal{H}$. Therefore, using matrix representation of the operator D_T^2, we have

$$\begin{bmatrix} I - T_1^*T_1 & -T_1^*T_2 \cdots & -T_1^*T_n \\ -T_2^*T_1 & I - T_2^*T_2 \cdots & -T_2^*T_n \\ \vdots & \vdots & \vdots \\ -T_n^*T_1 & -T_n^*T_2 \cdots & I - T_n^*T_n \end{bmatrix} \begin{bmatrix} \delta_{\alpha_1} \\ \delta_{\alpha_2} \\ \vdots \\ \delta_{\alpha_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$\delta_{\alpha_j} = \gamma_{\alpha - e_j} T_j^{\alpha_j - 1} T_1^{\alpha_1} \cdots T_n^{\alpha_n} D_{T^*},$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ with $\alpha \geq m+1$ and $j = 1, \ldots, n$. From the above identity, we have

$$\sum_{\substack{j=1 \\ j \neq i}}^n -T_i^*T_j \delta_{\alpha_j} + (I - T_i^*T_i)\delta_{\alpha_i} = 0,$$

and hence

$$\delta_{\alpha_i} = T_i^*\big(\sum_{j=1}^n T_j \delta_{\alpha_j}\big).$$

Replacing $\delta_{\alpha_j} = \gamma_{\alpha-e_j} T_j^{\alpha_j-1} T_1^{\alpha_1} \cdots T_n^{\alpha_n} D_{T^*}$ in the above identity, we get

$$\begin{aligned}\gamma_{\alpha-e_i} T_i^{\alpha_i-1} T_1^{\alpha_1} \cdots T_n^{\alpha_n} D_{T^*} &= T_i^* \big(\sum_{j=1}^{n} T_j (\gamma_{\alpha-e_j} T_j^{\alpha_j-1} T_1^{\alpha_1} \cdots T_n^{\alpha_n} D_{T^*}) \big) \\ &= \big[\sum_{j=1}^{n} \gamma_{\alpha-e_j} \big] (T_i^* T_1^{\alpha_1} \cdots T_n^{\alpha_n} D_{T^*}) \\ &= \big[\sum_{j=1}^{n} \gamma_{\alpha-e_j} \big] (T_i^* (T^\alpha D_{T^*})).\end{aligned}$$

Since $|\alpha| \geq m+1$, $\alpha_k \geq 1$ for some $k \in \{1, \ldots, n\}$. Therefore $\gamma_{\alpha-e_k} \neq 0$ for some $k \in \{1, \ldots, n\}$. Then

$$\begin{aligned}T_i^*(T^\alpha D_{T^*}) &= \frac{\gamma_{\alpha-e_i}}{\big[\sum_{j=1}^{n} \gamma_{\alpha-e_j} \big]} T_i^{\alpha_i-1} T_1^{\alpha_1} \cdots T_n^{\alpha_n} D_{T^*} \\ &= \frac{\gamma_{\alpha-e_i}}{\big[\sum_{j=1}^{n} \gamma_{\alpha-e_j} \big]} (T^{\alpha-e_i} D_{T^*}).\end{aligned}$$

Finally, since

$$\frac{\gamma_{\alpha-e_i}}{\big[\sum_{j=1}^{n} \gamma_{\alpha-e_j} \big]} = \frac{\alpha_i}{|\alpha|},$$

it follows that $T_i^*(T^\alpha D_{T^*}) = \frac{\alpha_i}{|\alpha|}(T^{\alpha-e_i} D_{T^*})$. ■

Lemma 3.2. *Let $T = (T_1, \ldots, T_n)$ be a commuting row contraction on a Hilbert space $\mathcal{H}$. If θ_T is a polynomial of degree m, then*

$$T^\alpha \mathcal{D}_{T^*} \perp T^\beta \mathcal{D}_{T^*},$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, $\alpha \neq \beta$ and $|\alpha|, |\beta| \geq m$.

Proof. If $\gamma \in \mathbb{Z}_+^n$, $|\gamma| \geq m$ and $i = 1, \ldots, n$, then by Lemma 3.1, we have

$$T^\gamma D_{T^*} = \frac{|\gamma|+1}{\gamma_i+1} T_i^* T_i T^\gamma D_{T^*}. \tag{3.1}$$

Now we fix $\alpha, \beta \in \mathbb{Z}_+^n$ such that $\alpha \neq \beta$ and $|\alpha|, |\beta| \geq m$. Since $\alpha \neq \beta$, $\alpha_j \neq \beta_j$ for some $j \in \{1, \ldots, n\}$. Without loss of generality, we assume that $\alpha_j < \beta_j$. Fix an integer $k \in \{1, \ldots, n\}$ such that $k \neq j$. By (3.1), we have

$$T^\beta D_{T^*} = c_k T_k^* T_k T^\beta D_{T^*},$$

where

$$c_k = \frac{|\beta|+1}{\beta_k+1}.$$

By repeated applications of (3.1), we have

$$T^\beta D_{T^*} = \big(c_k \cdots c_{k+m+1}\big) T_k^{*m+1} T_k^{m+1} T^\beta D_{T^*},$$

for some positive scalars $c_k, \ldots, c_{k+m+1}$. Hence for h_1 and h_2 in $\mathcal{H}$, we have

$$\langle T^\alpha D_{T^*} h_1, T^\beta D_{T^*} h_2\rangle = \big(c_k \cdots c_{k+m+1}\big)\langle T_k^{m+1} T^\alpha D_{T^*} h_1, T_k^{m+1} T^\beta D_{T^*} h_2\rangle,$$

where, on the other hand

$$\begin{aligned}\langle T_k^{m+1} T^\alpha D_{T^*} h_1, T_k^{m+1} T^\beta D_{T^*} h_2\rangle &= \langle T_j^{\alpha_j} T_k^{m+1}\Big(\prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) h_1, T_j^{\beta_j} T_k^{m+1}\Big(\prod_{i\neq j} T_i^{\beta_i} D_{T^*}\Big) h_2\rangle \\ &= \langle T_j^{*\beta_j} T_j^{\alpha_j} T_k^{m+1}\Big(\prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) h_1, T_k^{m+1}\Big(\prod_{i\neq j} T_i^{\beta_i} D_{T^*}\Big) h_2\rangle.\end{aligned}$$

But

$$\begin{aligned}T_j^{*\beta_j} T_j^{\alpha_j} T_k^{m+1}\Big(\prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) &= T_j^{*(\beta_j-1)}(T_j^* T_j)\Big(T_j^{\alpha_j-1} T_k^{m+1} \prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) \\ &= c T_j^{*(\beta_j-1)}\Big(T_j^{\alpha_j-1} T_k^{m+1} \prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big),\end{aligned}$$

for some positive scalar c, which follows from Lemma 3.1. By setting $\tilde{c} = c c_k \cdots c_{k+m+1}$, it follows that

$$\langle T^\alpha D_{T^*} h_1, T^\beta D_{T^*} h_2\rangle = \tilde{c}\langle T_j^{*(\beta_j-1)}\Big(T_j^{\alpha_j-1} T_k^{m+1} \prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) h_1, T_k^{m+1}\Big(\prod_{i\neq j} T_i^{\beta_i} D_{T^*}\Big) h_2\rangle.$$

Since $\beta_j > \alpha_j$, applying again Lemma 3.1 (possibly finitely many times), we get a constant $\hat{c}$ such that

$$T_j^{*(\beta_j-1)}\Big(T_j^{\alpha_j-1} T_k^{m+1} \prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) = \hat{c} T_j^{*(\beta_j-\alpha_j)}\Big(T_k^{m+1} \prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big),$$

and hence

$$\langle T^\alpha D_{T^*} h_1, T^\beta D_{T^*} h_2\rangle = \tilde{c}\hat{c}\langle T_j^*\Big(T_k^{m+1} \prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) h_1, T_j^{\beta_j-\alpha_j-1} T_k^{m+1}\Big(\prod_{i\neq j} T_i^{\beta_i} D_{T^*}\Big) h_2\rangle.$$

But once again, by Lemma 3.1, it follows that

$$T_j^*\Big(T_k^{m+1} \prod_{i\neq j} T_i^{\alpha_i} D_{T^*}\Big) = 0.$$

This implies that $\langle T^\alpha D_{T^*} h_1, T^\beta D_{T^*} h_2\rangle = 0$ and completes the proof of the lemma. ■

Now let T be an n-tuple of commuting row contraction on $\mathcal{H}$ such that the characteristic function θ_T is a polynomial of degree m. Set

$$\mathcal{M} = \overline{\text{span}}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, |\alpha| \geq m, \alpha \in \mathbb{Z}_+^n\},$$

and

$$\mathcal{N} = \overline{\text{span}}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, |\alpha| = m, \alpha \in \mathbb{Z}_+^n\}.$$

Clearly, $\mathcal{M}$ is a joint T-invariant subspace of $\mathcal{H}$ and $\mathcal{N} \subseteq \mathcal{M}$. Define

$$M_i := T_i|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M}) \qquad (i = 1, \ldots, n).$$

Then $(M_1, \ldots, M_n)$ is a commuting row contraction on $\mathcal{M}$. If $|\alpha| > m$, $\alpha \in \mathbb{Z}_+^n$, then Lemma 3.1 implies that

$$M_i M_i^*(T^\alpha D_{T^*}) = M_i T_i^* T^\alpha D_{T^*} = \frac{\alpha_i}{|\alpha|} M_i T^{\alpha - e_i} D_{T^*} = \frac{\alpha_i}{|\alpha|} T^\alpha D_{T^*},$$

for all $i = 1, \ldots, n$, and hence

$$\Big(\sum_{i=1}^n M_i M_i^*\Big)|_{\mathcal{M}\ominus\mathcal{N}} = I_{\mathcal{M}\ominus\mathcal{N}}.$$

Moreover, if $\beta \in \mathbb{Z}_+^n$ and $|\beta| = m$, then, again, Lemma 3.2 implies that

$$T_i^* T^\beta \mathcal{D}_{T^*} \perp T^\gamma D_{T^*},$$

for all $i = 1, \ldots, n$, and $\gamma \in \mathbb{Z}_+^n$ and $|\gamma| \geq m$. This implies that $M_i^*|_{\mathcal{N}} = 0$ for all $i = 1, \ldots, n$, and hence we find

$$I_{\mathcal{M}} - (M_1 M_1^* + \cdots + M_n M_n^*) = P_{\mathcal{N}}. \tag{3.2}$$

In particular, $\mathcal{N} = \mathcal{M} \ominus \big(\sum_{i=1}^n M_i \mathcal{M}\big)$. This also implies that the minimal closed joint $(M_1, \ldots, M_n)$-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$ is $\mathcal{M}$ itself. Moreover, by virtue of Lemma 3.1, it follows easily that $(M_1, \ldots, M_n)$ is a pure tuple. We summarize these observations as follows:

Theorem 3.3. *Let $T = (T_1, \ldots, T_n)$ be a commuting row contraction on $\mathcal{H}$. Assume that the characteristic function of T is a polynomial of degree m. If*

$$\mathcal{M} = \overline{span}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, |\alpha| \geq m, \alpha \in \mathbb{Z}_+^n\},$$

and $M_i := T_i|_{\mathcal{M}}$ for all $i = 1, \ldots, n$, and

$$\mathcal{N} = \overline{span}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, |\alpha| = m, \alpha \in \mathbb{Z}_+^n\},$$

then $\mathcal{M}$ is a joint closed invariant subspace for T and the restriction tuple $M = (M_1, \ldots, M_n)$ is a commuting pure partial isometry on $\mathcal{M}$. Moreover

$$\mathcal{M} = \overline{span}\{M^\alpha \mathcal{N} : \alpha \in \mathbb{Z}_+^n\},$$

and

$$\mathcal{N} = \mathcal{M} \ominus \big(\sum_{i=1}^n M_i \mathcal{M}\big),$$

and $\mathcal{M}$ is the minimal closed joint M-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$.

A priori, the above result suggests that the n-tuple M on $\mathcal{M}$, up to unitary equivalence, is just the multiplication tuple $(M_{z_1}, \ldots, M_{z_n})$ on $H_n^2(\mathcal{N})$, the $\mathcal{N}$-valued Drury-Arveson shift. It is also instructive to note that for $n = 1$ case [7] and for n-tuples of noncommutative operators [17], the operator M on $\mathcal{M}$ is indeed the multiplication operator or the tuple of creation operators on vector-valued Hardy space or the Fock space, respectively. However, for n-tuples of commuting row contractions, $n > 1$, this is not true in general. This problem is connected to Gleason's property (also known as Gleason's problem) of functions on the unit ball.

For the convenience of the reader, we recall *Gleason's problem* in the Drury-Arveson space. Let $\boldsymbol{w} \in \mathbb{B}^n$ and let $f \in H^2_n$. If $f(\boldsymbol{w}) = 0$, then the Gleason problem says that [1] there exist $f_1, \ldots, f_n \in H^2_n$ such that

$$f(\boldsymbol{z}) = \sum_{i=1}^{n} (z_i - w_i) f_i(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Then, in view of the fact that $(M_z - W)(H^2_n)^n$ is a closed subspace of H^2_n and

$$\bigcap_{i=1}^{n} \ker(M_{z_i} - w_i I_{H^2_n})^* = \mathbb{C}k(\cdot, \boldsymbol{w}),$$

it follows that

$$H^2_n = (M_z - W)(H^2_n)^n \dotplus \mathbb{C},$$

for all $\boldsymbol{w} \in \mathbb{B}^n$, where $\dotplus$ denotes the algebraic direct sum of subspaces. With this as motivation, we define regular tuples of operators [5, Section 2].

Definition 3.4. *We say that a tuple of commuting bounded linear operators $T = (T_1, \ldots, T_n)$ on a Hilbert space $\mathcal{H}$ is regular if there exists $\epsilon > 0$ such that for any $\|\boldsymbol{z}\|_{\mathbb{C}^n} < \epsilon$ the subspace $(T - Z)\mathcal{H}^n$ is closed in $\mathcal{H}$ and*

$$\mathcal{H} = (T - Z)\mathcal{H}^n \dotplus \Big(\mathcal{H} \ominus \sum_{i=1}^{n} T_i \mathcal{H}\Big).$$

Theorem 3.5. *In the setting of Theorem 3.3, if, in addition, the n-tuple M on $\mathcal{M}$ is regular, then M and the Drury-Arveson shift $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_n(\mathcal{N})$ are unitary equivalent.*

Proof. By (3.2), the tuple $M = (M_1, \ldots, M_n)$ on $\mathcal{M}$ satisfies $I_{\mathcal{M}} - MM^* = P_{\mathcal{N}}$, and hence M is a partial isometry and, in particular, $M^*M|_{\mathrm{ran}M^*} : \mathrm{ran}M^* \to \mathrm{ran}M^*$ is invertible. It follows that

$$(M^*M)|_{\mathrm{ran}M^*} = I_{\mathrm{ran}M^*},$$

and hence by [5, Theorem 3.5], the map

$$(Uf)(z) = \sum_{\alpha \in \mathbb{Z}^n_+} \gamma_\alpha \Big(P_{\mathcal{N}} M^{*\alpha} f\Big) z^\alpha,$$

defines a unitary operator $U : \mathcal{M} \to H^2_n(\mathcal{N})$ and satisfies $UM_i = M_{z_i}U$ for all $i = 1, \ldots, n$. ∎

We continue with the setting of Theorem 3.3, and define

$$\mathcal{K} = \overline{\mathrm{span}}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, \alpha \in \mathbb{Z}^n_+\},$$

and

$$\mathcal{H}_{\mathrm{nil}} = \mathcal{K} \ominus \mathcal{M} \quad \text{and} \quad N_i = P_{\mathcal{H}_{\mathrm{nil}}} T_i|_{\mathcal{H}_{\mathrm{nil}}},$$

for all $i = 1, \ldots, n$. Clearly, $\mathcal{H}_{\mathrm{nil}}$ is a semi-invariant subspace for T and hence

$$N^\alpha = P_{\mathcal{H}_{\mathrm{nil}}} T^\alpha|_{\mathcal{H}_{\mathrm{nil}}} \qquad (\alpha \in \mathbb{Z}^n_+).$$

In particular, $N_iN_j = N_jN_i$ for all $i, j = 1, \ldots, n$, and

$$\sum_{i=1}^{n} N_iN_i^* \leq P_{\mathcal{H}_{\mathrm{nil}}} \sum_{i=1}^{n} T_iT_i^*|_{\mathcal{H}_{\mathrm{nil}}} \leq I_{\mathcal{H}_{\mathrm{nil}}},$$

that is, $N = (N_1, \ldots, N_n)$ is a commuting row contraction on $\mathcal{H}_{\mathrm{nil}}$. Clearly $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq m$ implies $T^\alpha \mathcal{K} \subseteq \mathcal{M}$ and hence $N^\alpha = 0$. This shows that the commuting row contraction N is a nilpotent tuple of order $\leq m$. Moreover, we have

$$T_j|_{\mathcal{M}\oplus\mathcal{H}_{\mathrm{nil}}} = \begin{bmatrix} M_j & * \\ 0 & N_j \end{bmatrix} : \mathcal{M} \oplus \mathcal{H}_{\mathrm{nil}} \to \mathcal{M} \oplus \mathcal{H}_{\mathrm{nil}}.$$

Note now that $h \in \mathcal{H} \ominus \mathcal{K}$ if and only if $h \in \ker(D_{T^*}T^{*\alpha})$, or, equivalently, $h \in \ker(T^\alpha D_{T^*}^2 T^{*\alpha})$ for all $\alpha \in \mathbb{Z}_+^n$. Note also that

$$I - \sum_{|\alpha|=k} T^\alpha T^{*\alpha} = D_{T^*}^2 + (\sum_{|\beta|=1} T^\beta D_{T^*}^2 T^{*\beta}) + \cdots + (\sum_{|\beta|=k-1} T^\beta D_{T^*}^2 T^{*\beta}),$$

for all $k \geq 1$. This implies that $h \in \mathcal{H} \ominus \mathcal{K}$ if and only if h is in the right side of (2.2). Moreover, $\mathcal{H} \ominus \mathcal{K}$ is a T^*-invariant subspace of $\mathcal{H}$. Consequently

$$\mathcal{H}_c := \mathcal{H} \ominus \mathcal{K} = \{h \in \mathcal{H} : \sum_{|\alpha|=k} \|T^{*\alpha}h\|^2 = \|h\|^2 \text{ for all } k \in \mathbb{Z}_+\},$$

and $\sum_{i=1}^{n} W_iW_i^* = I_{\mathcal{H}_c}$, where $W_i = P_{\mathcal{H}_c}T_i|_{\mathcal{H}_c}$ for all $i = 1, \ldots, n$. Moreover

$$W_i^*W_j^* = (T_i^*|_{\mathcal{H}_c})(T_j^*|_{\mathcal{H}_c}) = T_i^*T_j^*|_{\mathcal{H}_c} = T_j^*T_i^*|_{\mathcal{H}_c} = W_j^*W_i^*,$$

for all $i, j = 1, \ldots, n$. It follows that W is a commuting spherical co-isometric tuple on $\mathcal{H}_c$. Recall that an n-tuple $(X_1, \ldots, X_n)$ on $\mathcal{L}$ is said to be a *spherical co-isometry* if $\sum_{i=1}^{n} X_iX_i^* = I_{\mathcal{L}}$.

Thus, we have proved:

THEOREM 3.6. *Let $T = (T_1, \ldots, T_n)$ be a commuting row contraction on a Hilbert space $\mathcal{H}$ with polynomial characteristic function of degree m. If $\mathcal{M} = \overline{span}\{T^\alpha D_{T^*}h : h \in \mathcal{H}, |\alpha| \geq m, \alpha \in \mathbb{Z}_+^n\}$, and*

$$\mathcal{H}_{nil} = \overline{span}\{T^\alpha D_{T^*}h : h \in \mathcal{H}, \alpha \in \mathbb{Z}_+^n\} \ominus \mathcal{M},$$

and

$$\mathcal{H}_c = \{h \in \mathcal{H} : \sum_{|\alpha|=k} \|T^{*\alpha}h\|^2 = \|h\|^2 \textit{ for all } k \in \mathbb{Z}_+\},$$

then $\mathcal{H} = \mathcal{M} \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$ and T_i, $i = 1, \ldots, n$ admits the following matrix decomposition

$$T_i = \begin{bmatrix} M_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}, \tag{3.3}$$

where M on $\mathcal{M}$ is a pure row contraction, N on $\mathcal{H}_{nil}$ is a commuting nilpotent tuple of order less than or equal to m and W on $\mathcal{H}_c$ is a commuting spherical co-isometry. Moreover,

$$\sum_{i=1}^{n} M_i M_i^* = I_{\mathcal{M}} - P_{\mathcal{N}},$$

where $\mathcal{N} = \mathcal{M} \ominus \Big(\sum_{i=1}^{n} T_i \mathcal{M}\Big)$. If, in addition, M is regular, then it is a Drury-Arveson shift.

In the final section, we will study a non-trivial example of pure partial isometric commuting tuple whose characteristic function is not a polynomial. Therefore it is not unitarily equivalent to a Drury Arveson shift. Because of Corollary 3.10 of [5], a pure partial isometric commuting tuple is unitarily equivalent to a Drury Arveson shift if and only if it is a regular tuple. Thus, the regularity assumption is an essential condition for the final conclusion in the above theorem.

For simplicity in what follows, we will refer to the representation (3.3) as simply the *canonical representation* of T with polynomial characteristic function of degree m. When the n-tuple M of the canonical representation of T is regular, we say that T is *regular*.

To avoid possible confusion, we remark in passing the following:

Remark 3.7. *If $m = 0$, then the above construction yields that $\mathcal{H}_{nil} = \{0\}$ and $N_i = 0$ for all $i = 1, \dots, n$.*

Remark 3.8. *Note that the first assertion of Theorem 3.6 appears to be similar to Popescu's result on noncommuting tuples* [17, Theorem 1.1]*. However, a closer look reveals that the noncommutative approach does not immediately work in the commutative setting. To be more specific, let $T = (T_1, \dots, T_n)$ be a commuting row contraction on $\mathcal{H}$. Suppose the noncommutative characteristic function Θ_T (see* section 4 *for the definition of noncommutative characteristic functions) is a noncommutative polynomial of degree m. Then by* [Theorem 1.1, [17]]*, there exist closed subspaces $\mathcal{H}_v, \mathcal{H}_{nil}$ and $\mathcal{H}_c$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$ and each $T_i, i = 1, \dots, n$, admits a representation*

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix},$$

where $V = (V_1, \dots, V_n)$ is a pure row isometry, $N = (N_1, \dots, N_n)$ is a nilpotent row contraction of order $\leq m$ on $\mathcal{H}_{nil}$, and $W = (W_1, \dots, W_n)$ is a coisometric row contraction on $\mathcal{H}_c$. But, in this representation, V is a commuting tuple as T is a commuting tuple. Then V is a commuting row isometry on $\mathcal{H}_v$ (that is, V_i is an isometry, $i = 1, \dots, n$, and $V_p(\mathcal{H}_v) \perp V_q(\mathcal{H}_v)$ for $p \neq q$). Therefore, if $n > 1$, then $V_i = 0$ for all $i = 1, \dots, n$, and $\mathcal{H}_v = \{0\}$. In particular, the first assertion of Theorem 3.6 does not follow from the representation of [Theorem 1.1, [17]]*. This view also applies to the results of* section 6.

4. Factorizations and noncommuting tuples

We now turn to characteristic functions of noncommuting tuples introduced by Popescu [11]. Here, following [8], we obtain an analytic structure of polynomial characteristic functions (up to unitary equivalence) of noncommuting row contractions.

The full *Fock space* over $\mathbb{C}^n$, denoted by Γ, is the Hilbert space

$$\Gamma := \bigoplus_{m=0}^{\infty} (\mathbb{C}^n)^{\otimes m} = \mathbb{C} \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^{\otimes 2} \oplus \cdots \oplus (\mathbb{C}^n)^{\otimes m} \oplus \cdots .$$

The *vacuum vector* $1 \oplus 0 \oplus \cdots \in \Gamma$ is denoted by $e_\emptyset$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$ and $\mathbb{F}_n^+$ be the unital free semi-group with generators $1, \ldots, n$ and the identity $\emptyset$. For $\alpha = \alpha_1 \cdots \alpha_m \in \mathbb{F}_n^+$ we denote the vector $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}$ by e_α. Then $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ forms an orthonormal basis of Γ. For each $j = 1, \ldots, n$, the *left creation operator* L_j and the *right creation operator* R_j on Γ are defined by

$$L_j f = e_j \otimes f, \qquad R_j f = f \otimes e_j \qquad (f \in \Gamma),$$

respectively. Moreover, $R_j = U^* L_j U$ where U, defined by

$$U(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}) = e_{i_m} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}, \tag{4.4}$$

is the *flip operator* on Γ. The *noncommutative disc algebra* $\mathcal{A}_n^\infty$ is the norm closed algebra generated by $\{I_\Gamma, L_1, \ldots, L_n\}$ and the *noncommutative analytic Toeplitz algebra* $\mathcal{F}_n^\infty$ is the WOT-closure of $\mathcal{A}_n^\infty$ (see Popescu [12]).

Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces and $M \in \mathcal{B}(\Gamma \otimes \mathcal{E}, \Gamma \otimes \mathcal{E}_*)$. Then M is said to be *multi-analytic operator* if

$$M(L_i \otimes I_{\mathcal{E}}) = (L_i \otimes I_{\mathcal{E}_*})M \qquad (i = 1, \ldots, n).$$

In this case, the bounded linear map $\theta \in \mathcal{B}(\mathcal{E}, \Gamma \otimes \mathcal{E}_*)$ defined by

$$\theta\eta = M(e_\emptyset \otimes \eta) \qquad (\eta \in \mathcal{E}),$$

is said to be the *symbol* of M and we denote $M = M_\theta$. Moreover, define $\theta_\alpha \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, $\alpha \in \mathbb{F}_n^+$ by

$$\langle \theta_\alpha \eta, \eta_* \rangle := \langle \theta\eta, e_{\bar\alpha} \otimes \eta_* \rangle = \langle M(e_\emptyset \otimes \eta), e_{\bar\alpha} \otimes \eta_* \rangle, \qquad (\eta \in \mathcal{E}, \eta_* \in \mathcal{E}_*)$$

where $\bar\alpha$ is the reverse of α. The Fourier type representation for multi-analytic operators was considered first by Popescu (see Popescu [18]), and from this representation, we have a unique formal Fourier expansion

$$M \sim \sum_{\alpha \in \mathbb{F}_n^+} R^\alpha \otimes \theta_\alpha,$$

and

$$M = \mathrm{SOT} - \lim_{r \to 1^-} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} R^\alpha \otimes \theta_\alpha$$

where $|\alpha|$ is the length of α. A multi-analytic operator $M_\theta \in \mathcal{B}(\Gamma \otimes \mathcal{E}, \Gamma \otimes \mathcal{E}_*)$ is said to be *purely contractive* if M_θ is a contraction and

$$\|P_{e_\emptyset \otimes \mathcal{E}_*} \theta\eta\| < \|\eta\| \qquad (\eta \in \mathcal{E}, \eta \neq 0).$$

We say that M_θ *coincides* with a multi-analytic operator $M_{\theta'} \in \mathcal{B}(\Gamma\otimes\mathcal{E}', \Gamma\otimes\mathcal{E}'_*)$ if there exist unitary operators $W : \mathcal{E} \to \mathcal{E}'$ and $W_* : \mathcal{E}_* \to \mathcal{E}'_*$ such that

$$(I_\Gamma \otimes W_*)M_\theta = M_{\theta'}(I_\Gamma \otimes W).$$

Let $\mathcal{H}$ be a Hilbert space and $T = (T_1, \dots, T_n)$ be a row operator on $\mathcal{H}$. For simplicity of the notations, we will denote by $\tilde{T}$ and $\tilde{R}$ the row operators $(I_\Gamma \otimes T_1, \dots, I_\Gamma \otimes T_n)$ and $(R_1 \otimes I_{\mathcal{H}}, \dots, R_n \otimes I_{\mathcal{H}})$ on $\Gamma \otimes \mathcal{H}$, respectively.

The *characteristic function* of a row contraction T on $\mathcal{H}$ is a purely contractive multi-analytic operator $\Theta_T \in \mathcal{B}(\Gamma \otimes \mathcal{D}_T, \Gamma \otimes \mathcal{D}_{T^*})$ defined by

$$\Theta_T \sim \; -I_\Gamma \otimes T + (I_\Gamma \otimes D_{T^*})(I_{\Gamma\otimes\mathcal{H}} - \tilde{R}\tilde{T}^*)^{-1}\tilde{R}(I_\Gamma \otimes D_T).$$

Hence

$$\Theta_T = \mathrm{SOT} - \lim_{r\to 1} \Theta_T(r\tilde{R}),$$

where for each $r \in [0, 1)$,

$$\Theta_T(r\tilde{R}) := -\tilde{T} + D_{\tilde{T}^*}(I_{\Gamma\otimes\mathcal{H}} - r\tilde{R}\tilde{T}^*)^{-1} r\tilde{R} D_{\tilde{T}}.$$

Therefore

$$\Theta_T = \mathrm{SOT} - \lim_{r\to 1} \Theta_T(r\tilde{R}) = \mathrm{SOT} - \lim_{r\to 1} \big[-\tilde{T} + D_{\tilde{T}^*}(I_{\Gamma\otimes\mathcal{H}} - r\tilde{R}\tilde{T}^*)^{-1} r\tilde{R} D_{\tilde{T}}\big]. \tag{4.5}$$

Now we recall the classical result of Sz.-Nagy and Foias concerning 2×2 block contractions (see [20], and also [6, Lemma 2.1, Chapter IV]):

THEOREM 4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $A = (A_1, \dots, A_n) \in \mathcal{B}(\mathcal{H}_1^n, \mathcal{H}_1)$, $B = (B_1, \dots, B_n) \in \mathcal{B}(\mathcal{H}_2^n, \mathcal{H}_2)$ and $X = (X_1, \dots, X_n) \in \mathcal{B}(\mathcal{H}_2^n, \mathcal{H}_1)$ be row operators. Then the row operator*

$$T = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1^n \oplus \mathcal{H}_2^n, \mathcal{H}_1 \oplus \mathcal{H}_2),$$

is a row contraction if and only if A and B are row contractions and $X = D_{A^*}LD_B$ for some contraction $L \in \mathcal{B}(\mathcal{D}_B, \mathcal{D}_{A^*})$.

Next, we recall a result [9, Theorem 2.2] concerning factorizations of characteristic functions of noncommutative tuples, which will be used in the proof of the main theorem of this section. Recall, given a contraction $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the *Julia-Halmos matrix* corresponding to L is defined by

$$J_L = \begin{bmatrix} L^* & D_L \\ D_{L^*} & -L \end{bmatrix}.$$

THEOREM 4.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Suppose A on $\mathcal{H}_1$ and B on $\mathcal{H}_2$ are n-tuples of row contractions and $L \in \mathcal{B}(\mathcal{D}_B, \mathcal{D}_{A^*})$ is a contraction, and let*

$$T = \begin{bmatrix} A & D_{A^*}LD_B \\ 0 & B \end{bmatrix} : \mathcal{H}_1^n \oplus \mathcal{H}_2^n \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Then there exist unitaries $\tau \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_A \oplus \mathcal{D}_L)$ and $\tau_* \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_{B^*} \oplus \mathcal{D}_{L^*})$ such that

$$\Theta_T = (I_\Gamma \otimes \tau_*^{-1}) \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes J_L) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_L} \end{bmatrix} (I_\Gamma \otimes \tau),$$

where $J_L \in \mathcal{B}(\mathcal{D}_{A^*} \oplus \mathcal{D}_L, \mathcal{D}_B \oplus \mathcal{D}_{L^*})$ is the Julia-Halmos matrix corresponding to L.

We are now ready to prove the main factorization result of this section. This is a non-commutative version of [8, Theorem 1.3]. The line of the proof also follows the idea of [8, Theorem 1.3].

THEOREM 4.3. *Let $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ be Hilbert spaces. Suppose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$ and assume that $T = (T_1, \dots, T_n)$ is a row contraction on $\mathcal{H}$ and*

$$T_i = \begin{bmatrix} S_i & * & * \\ 0 & N_i & * \\ 0 & 0 & C_i \end{bmatrix},$$

for all $i = 1, \dots, n$. Then S, N and C are n-tuples of row contractions on $\mathcal{H}_1, \mathcal{H}_0$ and $\mathcal{H}_{-1}$, respectively, and there exist Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}$, and unitary operators

$$\tau_1 \in \mathcal{B}(\mathcal{D}_{N^*} \oplus \mathcal{E}, \mathcal{D}_C \oplus \mathcal{E}_1) \quad \text{and} \quad \tau_2 \in B(\mathcal{D}_{S^*} \oplus \mathcal{E}_2, \mathcal{D}_N \oplus \mathcal{E}),$$

such that Θ_T coincides with

$$\begin{bmatrix} \Theta_C & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}_1} \end{bmatrix} (I_\Gamma \otimes \tau_1) \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}} \end{bmatrix} (I_\Gamma \otimes \tau_2) \begin{bmatrix} \Theta_S & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}_2} \end{bmatrix}.$$

Proof. For each $i = 1, \dots, n$, set $T_i = \begin{bmatrix} A_i & Y_i \\ 0 & C_i \end{bmatrix}$, where $A_i = \begin{bmatrix} S_i & X_i \\ 0 & N_i \end{bmatrix} = P_{\mathcal{H}_1 \oplus \mathcal{H}_0} T_i|_{\mathcal{H}_1 \oplus \mathcal{H}_0}$. Since T is a row contraction, by Theorem 4.1, A and C are row contractions and there exists a contraction $L_Y : \mathcal{D}_C \to \mathcal{D}_{A^*}$ such that $Y = D_{A^*} L_Y D_C$. On the other hand, since A is a row contraction, by Theorem 4.1 again, it follows that S and N are row contractions and $X = D_{S^*} L_X D_N$ for some contraction $L_X : \mathcal{D}_N \to \mathcal{D}_{S^*}$. Now, applying Theorem 4.2 to the row contraction $T = \begin{bmatrix} A & D_{A^*} L_Y D_C \\ 0 & C \end{bmatrix}$, we obtain

$$\Theta_T = (I_\Gamma \otimes u_*^{-1}) \begin{bmatrix} \Theta_C & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y^*}} \end{bmatrix} (I_\Gamma \otimes J_{L_Y}) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} (I_\Gamma \otimes u),$$

for some unitary operators $u \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_A \oplus \mathcal{D}_{L_Y})$ and $u_* \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_{C^*} \oplus \mathcal{D}_{L_Y^*})$. Note that $J_{L_Y} \in \mathcal{B}(\mathcal{D}_{A^*} \oplus \mathcal{D}_{L_Y}, \mathcal{D}_C \oplus \mathcal{D}_{L_Y^*})$ is the Julia-Halmos matrix corresponding to L_Y. Again, applying Theorem 4.2 to the row contraction $A = \begin{bmatrix} S & D_{S^*} L_X D_N \\ 0 & N \end{bmatrix}$, we obtain

$$\Theta_A = (I_\Gamma \otimes \sigma_*^{-1}) \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_X^*}} \end{bmatrix} (I_\Gamma \otimes J_{L_X}) \begin{bmatrix} \Theta_S & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_X}} \end{bmatrix} (I_\Gamma \otimes \sigma),$$

for some unitary operators $\sigma \in \mathcal{B}(\mathcal{D}_A, \mathcal{D}_S \oplus \mathcal{D}_{L_X})$ and $\sigma_* \in \mathcal{B}(\mathcal{D}_{A^*}, \mathcal{D}_{N^*} \oplus \mathcal{D}_{L_X^*})$. Again note that $J_{L_X} \in \mathcal{B}(\mathcal{D}_{S^*} \oplus \mathcal{D}_{L_X}, \mathcal{D}_N \oplus \mathcal{D}_{L_X^*})$ is the Julia-Halmos matrix corresponding to L_X. For convenience, we denote

$$\Phi_S = \begin{bmatrix} \Theta_S & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_X}} \end{bmatrix}, \Phi_N = \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_X^*}} \end{bmatrix}, \text{ and } \Phi_C = \begin{bmatrix} \Theta_C & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y^*}} \end{bmatrix}.$$

Therefore

$$\begin{aligned}
\Theta_T &= (I_\Gamma \otimes u_*^{-1})\Phi_C(I_\Gamma \otimes J_{L_Y}) \begin{bmatrix} (I_\Gamma \otimes \sigma_*^{-1})\Phi_N(I_\Gamma \otimes J_{L_X})\Phi_S(I_\Gamma \otimes \sigma) & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} (I_\Gamma \otimes u) \\
&= (I_\Gamma \otimes u_*^{-1})\Phi_C(I_\Gamma \otimes J_{L_Y}) \begin{bmatrix} (I_\Gamma \otimes \sigma_*^{-1}) & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} \begin{bmatrix} \Phi_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} \\
&\quad \times \begin{bmatrix} (I_\Gamma \otimes J_{L_X}) & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} \begin{bmatrix} \Phi_S & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} \begin{bmatrix} I_\Gamma \otimes \sigma & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} (I_\Gamma \otimes u) \\
&= (I_\Gamma \otimes u_*^{-1})\Phi_C(I_\Gamma \otimes \tau_1) \begin{bmatrix} \Phi_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} (I_\Gamma \otimes \tau_2) \begin{bmatrix} \Phi_S & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} (I_\Gamma \otimes v).
\end{aligned}$$

Here $\tau_1 \in \mathcal{B}((\mathcal{D}_{N^*} \oplus \mathcal{D}_{L_X^*}) \oplus \mathcal{D}_{L_Y}, \mathcal{D}_C \oplus \mathcal{D}_{L_Y^*})$, $\tau_2 \in \mathcal{B}((\mathcal{D}_{S^*} \oplus \mathcal{D}_{L_X}) \oplus \mathcal{D}_{L_Y}, (\mathcal{D}_N \oplus \mathcal{D}_{L_X^*}) \oplus \mathcal{D}_{L_Y})$ and $\psi \in \mathcal{B}(\mathcal{D}_T, (\mathcal{D}_S \oplus \mathcal{D}_{L_X}) \oplus \mathcal{D}_Y)$ are unitary operators defined by

$$I_\Gamma \otimes \tau_1 = (I_\Gamma \otimes J_{L_Y}) \begin{bmatrix} (I_\Gamma \otimes \sigma_*^{-1}) & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} \quad \text{and} \quad I_\Gamma \otimes \tau_2 = \begin{bmatrix} (I_\Gamma \otimes J_{L_X}) & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix},$$

and

$$I_\Gamma \otimes v = \begin{bmatrix} I_\Gamma \otimes \sigma & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L_Y}} \end{bmatrix} (I_\Gamma \otimes u).$$

Hence

$$\Theta_T = (I_\Gamma \otimes u_*^{-1}) \begin{bmatrix} \Theta_C & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}_1} \end{bmatrix} (I_\Gamma \otimes \tau_1) \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}} \end{bmatrix} (I_\Gamma \otimes \tau_2) \begin{bmatrix} \Theta_S & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}_2} \end{bmatrix} (I_\Gamma \otimes v),$$

where $\mathcal{E}_1 = \mathcal{D}_{L_Y^*}$, $\mathcal{E}_2 = \mathcal{D}_{L_X} \oplus \mathcal{D}_{L_Y}$ and $\mathcal{E} = \mathcal{D}_{L_X^*} \oplus \mathcal{D}_{L_Y}$. This completes the proof of the theorem. ∎

The following corollary is a noncommutative generalization of [8, Theorem 2.2].

COROLLARY 4.4. *Assume the setting of Theorem 4.3. If S is an isometry and C is a spherical co-isometry, then there exist a Hilbert space $\mathcal{E}$, a co-isometry $G_1 \in \mathcal{B}(\Gamma \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), \Gamma \otimes \mathcal{D}_{T^*})$ and an isometry $G_2 \in \mathcal{B}(\Gamma \otimes \mathcal{D}_T, \Gamma \otimes (\mathcal{D}_N \oplus \mathcal{E}))$ such that*

$$\Theta_T = G_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_\mathcal{E}} \end{bmatrix} G_2.$$

Proof. Since $\mathcal{D}_S = \{0_{\mathcal{H}_1^n}\}$ and $\mathcal{D}_{C^*} = \{0_{\mathcal{H}_{-1}}\}$, by assumption, it follows that the characteristic functions $\Theta_S : \Gamma \otimes \mathcal{D}_S \to \Gamma \otimes \mathcal{D}_{S^*}$ and $\Theta_C : \Gamma \otimes \mathcal{D}_C \to \Gamma \otimes \mathcal{D}_{C^*}$ are identically zero, that is,

$$0_S := \Theta_S \equiv 0 : \Gamma \otimes \{0_{\mathcal{H}_1^n}\} \to \Gamma \otimes \mathcal{D}_{S^*} \quad \text{and} \quad 0_C := \Theta_C \equiv 0 : \Gamma \otimes \mathcal{D}_C \to \Gamma \otimes \{0_{\mathcal{H}_{-1}}\}.$$

In this case, the unitary operators u_*, σ and v in the proof of Theorem 4.3 become

$$u_* \in \mathcal{B}(\mathcal{D}_{T^*}, \{0_{\mathcal{H}_{-1}}\} \oplus \mathcal{D}_{L_Y^*}) \quad \text{and} \quad \sigma \in \mathcal{B}(\mathcal{D}_A, \{0_{\mathcal{H}_1^n}\} \oplus \mathcal{D}_{L_X}),$$

and $v \in \mathcal{B}(\mathcal{D}_T, (\{0_{\mathcal{H}_1^n}\} \oplus \mathcal{D}_{L_X}) \oplus \mathcal{D}_Y)$, respectively. Then the representation of Θ_T, as given in the final part of the proof of Theorem 4.3, becomes

$$\Theta_T = G_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma\otimes(\mathcal{D}_{L_X^*}\oplus\mathcal{D}_{L_Y})} \end{bmatrix} G_2,$$

where $\mathcal{E} = \mathcal{D}_{L_X^*} \oplus \mathcal{D}_{L_Y}$ and

$$G_1 = (I_\Gamma \otimes u_*^{-1}) \begin{bmatrix} 0_C & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_{L_Y^*}} \end{bmatrix} (I_\Gamma \otimes \tau_1) \in \mathcal{B}(\Gamma \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), \Gamma \otimes \mathcal{D}_{T^*}),$$

and

$$G_2 = (I_\Gamma \otimes \tau_2) \begin{bmatrix} 0_S & 0 \\ 0 & I_{\Gamma\otimes(\mathcal{D}_{L_X}\oplus\mathcal{D}_{L_Y})} \end{bmatrix} (I_\Gamma \otimes v) \in \mathcal{B}(\Gamma \otimes \mathcal{D}_T, \Gamma \otimes (\mathcal{D}_N \oplus \mathcal{E})).$$

Since $0_C 0_C^* = I_{\Gamma\otimes\{0_{\mathcal{H}_{-1}}\}}$ and $0_S^* 0_S = I_{\Gamma\otimes\{0_{\mathcal{H}_1^n}\}}$, it follows that $G_1 G_1^* = I_{\Gamma\otimes\mathcal{D}_{T^*}}$ and $G_2^* G_2 = I_{\Gamma\otimes\mathcal{D}_T}$. This completes the proof. ∎

In view of Popescu [17, Theorem 1.1], the following corollary is now more definite:

Corollary 4.5. *Let T be a row contraction on $\mathcal{H}$ such that the characteristic function Θ_T is a noncommutative polynomial of degree m. Then there exist a Hilbert space $\mathcal{E}$, a nilpotent row contraction $N = (N_1, \ldots, N_n)$ of order $\leq m$, such that*

$$\Theta_T = G_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_\mathcal{E}} \end{bmatrix} G_2,$$

where G_1 and G_2 are co-isometry and isometry in $\mathcal{B}(\Gamma \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), \Gamma \otimes \mathcal{D}_{T^*})$ and $\mathcal{B}(\Gamma \otimes \mathcal{D}_T, \Gamma \otimes (\mathcal{D}_N \oplus \mathcal{E}))$, respectively.

Proof. Since T is a row contraction with a noncommutative polynomial of degree m, by [17, Theorem 1.1], there exist closed subspaces $\mathcal{H}_{-1}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ of $\mathcal{H}$ such $\mathcal{H} = \mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1$ and the matrix representation of T_i is given by

$$T_i = \begin{bmatrix} S_i & * & * \\ 0 & N_i & * \\ 0 & 0 & C_i \end{bmatrix},$$

for all $i = 1, \ldots, n$, where S is a row isometry on $\mathcal{H}_1$, N is a nilpotent row contraction of order $\leq m$ on $\mathcal{H}_0$ and C is a spherical co-isometry on $\mathcal{H}_{-1}$. The remaining part of the proof now directly follows from Corollary 4.4. ∎

5. Constrained Row Contractions and Factorizations

In this section, we describe the factorization results obtained in the previous section in the setting of constrained row contractions. Constrained row contractions are related to the notion of noncommutative varieties, which was introduced by G. Popescu in [13]. The added complications and structures of noncommutative varieties are due mostly to the fact, as for example, that the Drury-Arveson space is a quotient space of the full Fock space (see [13, 16]).

First, we recall the basic features of constrained row contractions and noncommutative varieties and refer the reader to Popescu [13, 16] for further details.

Let $\mathcal{P}_J \subset \mathcal{F}_n^\infty$ be a set of noncommutative polynomials, and let J be the WOT-closed two sided ideal of $\mathcal{F}_n^\infty$ generated by $\mathcal{P}_J$. In what follows, we always assume that $J \neq \mathcal{F}_n^\infty$. Then

$$\mathcal{M}_J := \overline{\text{span}}\{\phi \otimes \psi : \phi \in J, \psi \in \Gamma\} \quad \text{and} \quad \mathcal{N}_J := \Gamma \ominus \mathcal{M}_J$$

are proper joint $(L_1, \ldots, L_n)$ and $(L_1^*, \ldots, L_n^*)$ invariant subspaces of Γ, respectively. Define *constrained left creation operators* and *constrained right creation operators* on $\mathcal{N}_J$ by

$$V_j := P_{\mathcal{N}_J} L_j|_{P_{\mathcal{N}_J}} \qquad \text{and} \qquad W_j := P_{\mathcal{N}_J} R_j|_{P_{\mathcal{N}_J}} \qquad (j = 1, \ldots, n),$$

respectively. Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces, and let $M \in \mathcal{B}(\mathcal{N}_J \otimes \mathcal{E}, \mathcal{N}_J \otimes \mathcal{E}_*)$. Then M is said to be *constrained multi-analytic operator* if

$$M(V_j \otimes I_{\mathcal{E}}) = (V_j \otimes I_{\mathcal{E}_*})M \qquad (j = 1, \ldots, n).$$

A constrained multi-analytic operator $M \in \mathcal{B}(\mathcal{N}_J \otimes \mathcal{E}, \mathcal{N}_J \otimes \mathcal{E}_*)$ is said to be *purely contractive* if M is a contraction, $e_\emptyset \in \mathcal{N}_J$ and

$$\|P_{e_\emptyset \otimes \mathcal{E}_*} M(e_\emptyset \otimes \eta)\| < \|\eta\| \qquad (\eta \neq 0, \eta \in \mathcal{E}).$$

Let $\mathcal{W}(W_1, \ldots, W_n)$ denote the WOT-closed algebra generated by $\{I, W_1, \ldots, W_n\}$ and $\mathcal{R}_n^\infty = U^* \mathcal{F}_n^\infty U$, where U is the flipping operator (see (4.4)). The following equality, due to Popescu [13], is often useful:

$$\mathcal{W}(W_1, \ldots, W_n) \,\bar{\otimes}\, \mathcal{B}(\mathcal{E}, \mathcal{E}_*) = P_{\mathcal{N}_J \otimes \mathcal{E}_*}[\mathcal{R}_n^\infty \,\bar{\otimes}\, \mathcal{B}(\mathcal{E}, \mathcal{E}_*)]|_{P_{\mathcal{N}_J \otimes \mathcal{E}}}.$$

Recall also that a row contraction $T = (T_1, \ldots, T_n)$ on $\mathcal{H}$ is said to be *J-constrained row contraction*, or simply *constrained row contraction* if J is clear from the context, if

$$p(T_1, \ldots, T_n) = 0 \qquad (p \in \mathcal{P}_J).$$

The *constrained characteristic function* $\Theta_{J,T}$ (see Popescu [13]) of a constrained row contraction $T = (T_1, \ldots, T_n)$ on a Hilbert space $\mathcal{H}$ is defined by

$$\Theta_{J,T} = P_{\mathcal{N}_J \otimes \mathcal{D}_{T^*}} \Theta_T|_{\mathcal{N}_J \otimes \mathcal{D}_T}.$$

Note that $\Theta_{J,T}$ is a pure constrained multi-analytic operator $\Theta_{J,T} : \mathcal{N}_J \otimes \mathcal{D}_T \to \mathcal{N}_J \otimes \mathcal{D}_{T^*}$. Moreover, since $\mathcal{N}_J \otimes \mathcal{D}_{T^*}$ is a joint $(R_1^* \otimes I_{\mathcal{D}_{T^*}}, \ldots, R_n^* \otimes I_{\mathcal{D}_{T^*}})$ invariant subspace and $W_i \otimes I_{\mathcal{D}_{T^*}} = (P_{\mathcal{N}_J} R_i|_{P_{\mathcal{N}_J}}) \otimes I_{\mathcal{D}_{T^*}}$, $i = 1, \ldots, n$, it follows that (see [13])

$$\Theta_T^*(\mathcal{N}_J \otimes \mathcal{D}_{T^*}) \subset \mathcal{N}_J \otimes \mathcal{D}_T \quad \text{and} \quad \Theta_T(\mathcal{M}_J \otimes \mathcal{D}_T) \subset \mathcal{M}_J \otimes \mathcal{D}_{T^*}. \tag{5.6}$$

The starting point of our consideration of constrained row contractions is the following result [9, Theorem 3.1]:

THEOREM 5.1. *Let A on $\mathcal{H}_1$ and B on $\mathcal{H}_2$ be n-tuples of row contractions, and let $L \in \mathcal{B}(\mathcal{D}_B, \mathcal{D}_{A^*})$ be a contraction. If $T = \begin{bmatrix} A & D_{A^*}LD_B \\ 0 & B \end{bmatrix}$ is a constrained row contraction on $\mathcal{H}_1 \oplus \mathcal{H}_2$, then A and B are also constrained row contractions and there exist unitary operators $\sigma \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_A \oplus \mathcal{D}_L)$ and $\sigma_* \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_{B^*} \oplus \mathcal{D}_{L^*})$ such that*

$$\Theta_{J,T} = (I_{\mathcal{N}} \otimes \sigma_*^{-1}) \begin{bmatrix} \Theta_{J,B} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L^*}} \end{bmatrix} (I_{\mathcal{N}} \otimes J_L) \begin{bmatrix} \Theta_{J,A} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_L} \end{bmatrix} (I_{\mathcal{N}} \otimes \sigma)$$

where $J_L \in \mathcal{B}(\mathcal{D}_{A^*} \oplus \mathcal{D}_L, \mathcal{D}_B \oplus \mathcal{D}_{L^*})$ is the Julia-Halmos matrix corresponding to L.

We are now ready to prove the factorization result for constrained row contractions. However, the proof is similar in spirit to that of Theorem 4.3, and thus, we only sketch it.

THEOREM 5.2. *Let $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ be Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$. Suppose*

$$T_i = \begin{bmatrix} S_i & * & * \\ 0 & N_i & * \\ 0 & 0 & C_i \end{bmatrix} \qquad (i = 1, \ldots, n).$$

If T is a constrained row contraction, then S, N and C are also constrained row contractions on $\mathcal{H}_1, \mathcal{H}_0$ and $\mathcal{H}_{-1}$, respectively, and there exist Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}$ and unitary operators $\tau_1 \in \mathcal{B}(\mathcal{D}_{N^*} \oplus \mathcal{E}, \mathcal{D}_C \oplus \mathcal{E}_1)$ and $\tau_2 \in B(\mathcal{D}_{S^*} \oplus \mathcal{E}_2, \mathcal{D}_N \oplus \mathcal{E})$ such that $\Theta_{J,T}$ coincides with

$$\begin{bmatrix} \Theta_{J,C} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{E}_1} \end{bmatrix} (I_{\mathcal{N}} \otimes \tau_1) \begin{bmatrix} \Theta_{J,N} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{E}} \end{bmatrix} (I_{\mathcal{N}} \otimes \tau_2) \begin{bmatrix} \Theta_{J,S} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{E}_2} \end{bmatrix}.$$

Proof. We use the same notations as in the proof of Theorem 4.3: $T_i = \begin{bmatrix} A_i & Y_i \\ 0 & C_i \end{bmatrix}$, where $A_i = \begin{bmatrix} S_i & X_i \\ 0 & N_i \end{bmatrix} = P_{\mathcal{H}_1\oplus\mathcal{H}_0} T_i|_{\mathcal{H}_1\oplus\mathcal{H}_0}$ for all $i = 1, \ldots, n$. By Theorem 4.1 and first part of Theorem 5.1, we already know that A and C are constrained row contractions and $Y = D_{A^*} L_Y D_C$ for some contraction $L_Y : \mathcal{D}_C \to \mathcal{D}_{A^*}$. Repeating the argument to the constrained row contraction A, we obtain that S and N are also constrained row contractions and $X = D_{S^*} L_X D_N$ for some contraction $L_X : \mathcal{D}_N \to \mathcal{D}_{S^*}$. Then, applying Theorem 5.1 to the constrained row contractions $T = \begin{bmatrix} A & D_{A^*}L_Y D_C \\ 0 & C \end{bmatrix}$ and $A = \begin{bmatrix} S & D_{S^*}L_X D_N \\ 0 & N \end{bmatrix}$, we find

$$\Theta_{J,T} = (I_{\mathcal{N}} \otimes u_*^{-1}) \begin{bmatrix} \Theta_{J,C} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L_Y^*}} \end{bmatrix} (I_{\mathcal{N}} \otimes J_{L_Y}) \begin{bmatrix} \Theta_{J,A} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L_Y}} \end{bmatrix} (I_{\mathcal{N}} \otimes u),$$

for some unitary operators $u \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_A \oplus \mathcal{D}_{L_Y})$ and $u_* \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_{C^*} \oplus \mathcal{D}_{L_Y^*})$, and

$$\Theta_{J,A} = (I_{\mathcal{N}} \otimes \sigma_*^{-1}) \begin{bmatrix} \Theta_{J,N} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L_X^*}} \end{bmatrix} (I_{\mathcal{N}} \otimes J_{L_X}) \begin{bmatrix} \Theta_{J,S} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L_X}} \end{bmatrix} (I_{\mathcal{N}} \otimes \sigma),$$

for some unitary operators $\sigma \in \mathcal{B}(\mathcal{D}_A, \mathcal{D}_S \oplus \mathcal{D}_{L_X})$ and $\sigma_* \in \mathcal{B}(\mathcal{D}_{A^*}, \mathcal{D}_{N^*} \oplus \mathcal{D}_{L_X^*})$. Finally, by the same reasoning as in the proof of Theorem 4.3, it follows that

$$\Theta_{J,T} = (I_{\mathcal{N}} \otimes u_*^{-1}) \begin{bmatrix} \Theta_{J,C} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{E}_1} \end{bmatrix} (I_{\mathcal{N}} \otimes \tau_1) \begin{bmatrix} \Theta_{J,N} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{E}} \end{bmatrix} (I_{\mathcal{N}} \otimes \tau_2) \begin{bmatrix} \Theta_{J,S} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{E}_2} \end{bmatrix} (I_{\mathcal{N}} \otimes v),$$

where $\mathcal{E}_1 = \mathcal{D}_{L_Y^*}$, $\mathcal{E}_2 = \mathcal{D}_{L_X} \oplus \mathcal{D}_{L_Y}$ and $\mathcal{E} = \mathcal{D}_{L_X^*} \oplus \mathcal{D}_{L_Y}$, and $\tau_1 : (\mathcal{D}_{N^*} \oplus \mathcal{D}_{L_X^*}) \oplus \mathcal{D}_{L_Y} \to \mathcal{D}_C \oplus \mathcal{D}_{L_Y^*}$, $\tau_2 : (\mathcal{D}_{S^*} \oplus \mathcal{D}_{L_X}) \oplus \mathcal{D}_{L_Y} \to (\mathcal{D}_N \oplus \mathcal{D}_{L_X^*}) \oplus \mathcal{D}_{L_Y}$ and $v : \mathcal{D}_T \to (\mathcal{D}_S \oplus \mathcal{D}_{L_X}) \oplus \mathcal{D}_{L_Y}$ are unitary operators defined by

$$I_{\mathcal{N}} \otimes \tau_1 = (I_{\mathcal{N}} \otimes J_{L_Y}) \begin{bmatrix} (I_{\mathcal{N}} \otimes \sigma_*^{-1}) & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L_Y}} \end{bmatrix} \quad \text{and} \quad I_{\mathcal{N}} \otimes \tau_2 = \begin{bmatrix} (I_{\mathcal{N}} \otimes J_{L_X}) & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L_Y}} \end{bmatrix},$$

and

$$I_{\mathcal{N}} \otimes v = \begin{bmatrix} I_{\mathcal{N}} \otimes \sigma & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L_Y}} \end{bmatrix} (I_{\mathcal{N}} \otimes u).$$

This completes the proof of the theorem. ∎

The particular case of constrained row contractions where the noncommutative variety is given by

$$\mathcal{P}_{J_c} = \{L_iL_j - L_jL_i : i, j = 1, \dots, n\},$$

gives rise to commuting row contractions on Hilbert spaces. In this case, $\mathcal{N}_{J_c}$ becomes the symmetric Fock space Γ_s and the n-tuple V on Γ_s, where $V_j = P_{\Gamma_s} L_j|_{\Gamma_s}$, $j = 1, \dots, n$, becomes the left creation operators on Γ_s (see [4, 13],). More specifically, V on $\mathcal{N}_{J_c}$ and $(M_{z_1}, \dots, M_{z_n})$ on H^2_n are unitarily equivalent, where H^2_n is the Drury-Arveson space and M_{z_i} is the multiplication operator by the coordinate function z_i on H^2_n, $i = 1, \dots, n$. Under this identification, $P_{\Gamma_s} \mathcal{F}_n^\infty|_{\Gamma_s}$ corresponds to $\mathcal{M}(H^2_n)$, the multiplier algebra of H^2_n (see also Section 2).

From this point of view, if T on $\mathcal{H}$ is a constrained row contraction corresponding to $\mathcal{P}_{J_c}$, then $T_iT_j = T_jT_i$ for all $i, j = 1, \dots, n$, and one can identify the constrained characteristic function $\Theta_{J_c,T} = P_{\mathcal{N}_{J_c}\otimes\mathcal{D}_{T^*}} \Theta_T|_{\mathcal{N}_{J_c}\otimes\mathcal{D}_T}$ with the $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$-valued multiplier $\theta_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ in $\mathcal{M}(\mathcal{D}_T, \mathcal{D}_{T^*})$ [3, 4, 13], the characteristic function of the commuting tuple T (see (2.1)). In the remaining part of this paper, the identification of $\Theta_{J_c,T}$ and θ_T will be used interchangeably.

The first half of the following theorem is essentially a particular (the commutative) case of Theorem 5.2. The partially isometric property of V_2 in the remaining part is a special feature of n-tuples, $n > 1$, of commuting row contractions. The $n = 1$ case recovers [8, Theorem 1.3] with a proof essentially along the same line.

THEOREM 5.3. *Let $\mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ be Hilbert spaces, and let $T = (T_1, \dots, T_n)$ be an n-tuple of commuting row contraction on $\mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$ such that each T_i has the following matrix representation*

$$T_i = \begin{bmatrix} S_i & * & * \\ 0 & N_i & * \\ 0 & 0 & C_i \end{bmatrix} \qquad (i = 1, \dots, n).$$

with respect to $\mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$. *Then* S *on* $\mathcal{H}_1$, N *on* $\mathcal{H}_0$ *and* C *on* $\mathcal{H}_{-1}$ *are commuting row contractions, and there exist Hilbert spaces* $\mathcal{E}_1, \mathcal{E}_2$ *and* $\mathcal{E}$, *and unitary operators* $U_1 \in \mathcal{B}(\mathcal{D}_{N^*} \oplus \mathcal{E}, \mathcal{D}_C \oplus \mathcal{E}_1)$ *and* $U_2 \in B(\mathcal{D}_{S^*} \oplus \mathcal{E}_2, \mathcal{D}_N \oplus \mathcal{E})$ *such that* θ_T *coincides with*

$$\begin{bmatrix} \theta_C & 0 \\ 0 & I_{H^2_n \otimes \mathcal{E}_1} \end{bmatrix} (I_{H^2_n} \otimes U_1) \begin{bmatrix} \theta_N & 0 \\ 0 & I_{H^2_n \otimes \mathcal{E}} \end{bmatrix} (I_{H^2_n} \otimes U_2) \begin{bmatrix} \theta_S & 0 \\ 0 & I_{H^2_n \otimes \mathcal{E}_2} \end{bmatrix}.$$

In addition, if S *and* C *are Drury-Arveson shift and spherical co-isometry, respectively, then there exist a Hilbert space* $\mathcal{E}$, *a co-isometry* $G_1 \in (H^2_n \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), H^2_n \otimes \mathcal{D}_{T^*})$ *and a partial isometry* $G_2 \in (H^2_n \otimes \mathcal{D}_T, H^2_n \otimes (\mathcal{D}_N \oplus \mathcal{E}))$ *such that*

$$\theta_T = G_1 \begin{bmatrix} \theta_N & 0 \\ 0 & I_{\Gamma \otimes \mathcal{E}} \end{bmatrix} G_2.$$

Proof. We only need to prove the second half. Since C is a spherical co-isometry, $D_{C^*} = 0$, and hence $\mathcal{D}_{C^*} = \{0_{\mathcal{H}_{-1}}\}$. On the other hand, since S is the Drury-Arveson shift, θ_S is identically zero [15, Proposition 2.6]. Therefore, the unitary operators u_*, σ and v and the characteristic function $\Theta_{J,T}$, in terms of θ_T, in the proof of Theorem 5.2 becomes

$$u_* \in \mathcal{B}(\mathcal{D}_{T^*}, \{0_{\mathcal{H}_{-1}}\} \oplus \mathcal{D}_{L_Y^*}),\ \sigma \in \mathcal{B}(\mathcal{D}_A, \mathcal{D}_S \oplus \mathcal{D}_{L_X}),\ \text{and } v \in \mathcal{B}(\mathcal{D}_T, (\mathcal{D}_S \oplus \mathcal{D}_{L_X}) \oplus \mathcal{D}_{L_Y}),$$

and

$$\theta_T = G_1 \begin{bmatrix} \theta_N & 0 \\ 0 & I_{H^2_n \otimes (\mathcal{D}_{L_X^*} \oplus \mathcal{D}_{L_Y})} \end{bmatrix} G_2,$$

respectively, where $\mathcal{E} = \mathcal{D}_{L_X^*} \oplus \mathcal{D}_{L_Y}$,

$$G_1 = (I_{H^2_n} \otimes u_*^{-1}) \begin{bmatrix} 0_C & 0 \\ 0 & I_{H^2_n} \otimes \mathcal{D}_{L_Y^*} \end{bmatrix} (I_{H^2_n} \otimes U_1) \in \mathcal{B}(H^2_n \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), H^2_n \otimes \mathcal{D}_{T^*}),$$

$$G_2 = (I_{H^2_n} \otimes U_2) \begin{bmatrix} 0_S & 0 \\ 0 & I_{H^2_n \otimes (\mathcal{D}_{L_X} \oplus \mathcal{D}_{L_Y})} \end{bmatrix} (I_{H^2_n} \otimes v) \in \mathcal{B}(H^2_n \otimes \mathcal{D}_T, H^2_n \otimes (\mathcal{D}_N \oplus \mathcal{E})).$$

Since $0_C 0_C^* = I_{H^2_n \otimes \{0_{\mathcal{H}_{-1}}\}}$ and $0_S \equiv 0$, it is clear that G_1 is an co-isometry and G_2 is a partial isometry. ∎

If N on $\mathcal{H}_0$ is nilpotent of order m, then the above result clearly yields that the characteristic function θ_T is a polynomial of degree $\leq m$. Moreover, in view of Theorem 3.6, we have the following:

THEOREM 5.4. *Let* $T = (T_1, \ldots, T_n)$ *be an* n*-tuple of commuting row contraction on a Hilbert space* $\mathcal{H}$ *such that* θ_T *is a polynomial of degree* m, *and let*

$$\mathcal{M} = \overline{Span}\{T^\alpha D_{T^*} h : h \in \mathcal{H}, |\alpha| \geq m, \alpha \in \mathbb{Z}^n_+\}.$$

If T *is regular, then there exist a Hilbert space* $\mathcal{E}$, *a co-isometry* $G_1 \in \mathcal{B}(H^2_n \otimes (\mathcal{D}_{N^*} \oplus \mathcal{E}), H^2_n \otimes \mathcal{D}_{T^*})$ *and a partial isometry* $G_2 \in \mathcal{B}(H^2_n \otimes \mathcal{D}_T, H^2_n \otimes (\mathcal{D}_N \oplus \mathcal{E}))$ *such that*

$$\theta_T = G_1 \begin{bmatrix} \theta_N & 0 \\ 0 & I_{H^2_n \otimes \mathcal{D}_{\mathcal{E}}} \end{bmatrix} G_2.$$

Note that if $n = 1$, then the partial isometry G_2 becomes an isometry (see [8, Theorem 2.2]).

6. Uniqueness of the canonical representations

Recall that the canonical representation of a commuting row contraction T with polynomial characteristic function of degree m is the upper triangular representation of T_i on $\mathcal{H} = \mathcal{M} \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$ as in Theorem 3.6. In this section, we analyze the structure of canonical representations of commuting row contractions with polynomial characteristic functions.

Here also, some of our results are analogous to Popescu's noncommuting tuples of operators [17]. However, in view of Remark 3.8, our results are new and does not follow from Popescu. In fact, results of this section relies on the representations of tuples obtained in Theorem 3.6.

We first prove that $\mathcal{M}$ and $\mathcal{H}_c$ of the canonical representation are optimal in an appropriate sense (see [17, Proposition 1.4] for n-tuples of noncommutative row contractions).

Theorem 6.1. *Let T be an n-tuple of commuting row contraction on a Hilbert space $\mathcal{H}$ such that θ_T is a polynomial of degree m and also T is regular. Suppose $\mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ are Hilbert spaces, and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$. If the matrix representation of T_i with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$ is given by*

$$T_i = \begin{bmatrix} M_i^{'} & * & * \\ 0 & N_i^{'} & * \\ 0 & 0 & W_i^{'} \end{bmatrix} \qquad (i = 1, \ldots, n),$$

where M' on $\mathcal{H}_1$ is a Drury-Arveson shift, N' on $\mathcal{H}_0$ is nilpotent of order m and $W^{'}$ on $\mathcal{H}_{-1}$ is a spherical co-isometry, then $\mathcal{M} \subseteq \mathcal{H}_1$ and $\mathcal{H}_c \supseteq \mathcal{H}_{-1}$.

Proof. Since W' is a spherical co-isometry, with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$, we have $D_{T^*}^2 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{bmatrix}$. If $D_{T^*} = \begin{bmatrix} * & * & A_{31} \\ * & * & A_{32} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, then $D_{T^*}^2 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & A_{31}A_{31}^* + A_{32}A_{32}^* + A_{33}A_{33}^* \end{bmatrix}$, and hence $A_{31}A_{31}^* + A_{32}A_{32}^* + A_{33}A_{33}^* = 0$. It follows that $A_{31} = A_{32} = A_{33} = 0$. Therefore $D_{T^*} = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and hence on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$, we have

$$T^\alpha D_{T^*} = \begin{bmatrix} M'^\alpha & * & * \\ 0 & 0 & * \\ 0 & 0 & W'^\alpha \end{bmatrix} \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $T^\alpha D_{T^*}\mathcal{H} \subseteq \mathcal{H}_1$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq m$. Then, $\mathcal{M} \subseteq \mathcal{H}_1$, where, on the other hand, $\mathcal{H}_{-1} \subseteq \mathcal{H}_c$ as $\mathcal{H}_c$ is the maximal closed joint T^* invariant subspace of $\mathcal{H}$ such that $\begin{bmatrix} T_1^*|_{\mathcal{H}_c} \\ \vdots \\ T_n^*|_{\mathcal{H}_c} \end{bmatrix}$ is an isometry. ∎

Now we prove that the diagonal entries of the canonical representation of T, as in Theorem 3.6, is a complete unitary invariant. The noncommutative version of this is due to Popescu [17, Proposition 2.1].

PROPOSITION 6.2. *Let T on $\mathcal{H}$ and T' on $\mathcal{H}'$ be n-tuples of commuting row contractions with polynomial characteristic functions of degree m. Assume that*

$$T_i = \begin{bmatrix} M_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix} \text{ and } T_i^{'} = \begin{bmatrix} M_i^{'} & * & * \\ 0 & N_i^{'} & * \\ 0 & 0 & W_i^{'} \end{bmatrix} \qquad (i = 1, \dots, n),$$

are the canonical representations of T and T' on $\mathcal{H} = \mathcal{M} \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$ and $\mathcal{H}^{'} = \mathcal{M}^{'} \oplus \mathcal{H}_{nil}^{'} \oplus \mathcal{H}_c^{'}$ respectively. If $U : \mathcal{H} \to \mathcal{H}^{'}$ is a unitary operator such that $UT_i = T_i^{'}U$ for all $i = 1, \dots, n$, then $U\mathcal{M} = \mathcal{M}^{'}$, $U\mathcal{H}_{nil} = \mathcal{H}_{nil}^{'}$ and $U\mathcal{H}_c = \mathcal{H}_c^{'}$, and $(U|_{\mathcal{M}})M_i = M_i^{'}(U|_{\mathcal{M}})$, $(U|_{\mathcal{H}_{nil}})N_i = N_i^{'}(U|_{\mathcal{H}_{nil}})$ and $(U|_{\mathcal{H}_c})W_i = W_i^{'}(U|_{\mathcal{H}_c})$ for all $i = 1, \dots, n$.

Proof. Clearly, $UT_j = T_j{'}U$ and $UT_j^* = T_j^{'*}U$ for $j = 1, \dots, n$, implies that $UT^{\alpha}D_{T^*} = T^{'\alpha}D_{T^{'*}}U$, $\alpha \in \mathbb{Z}_+^n$, and, on the other hand, we have by definition $U\mathcal{M} = \mathcal{M}^{'}$. Moreover, since

$$\|T^{'*\alpha}(Uh)\|^2 = \|UT^{*\alpha}h\|^2 = \|T^{*\alpha}h\|^2,$$

for all $\alpha \in \mathbb{Z}_+^n$ and $h \in \mathcal{H}_c$, it follows that $U\mathcal{H}_c = \mathcal{H}_c^{'}$, and hence $U\mathcal{H}_{\text{nil}} = \mathcal{H}_{\text{nil}}^{'}$. The remaining part now follows from the representation $U = \begin{bmatrix} U|_{\mathcal{M}} & 0 & 0 \\ 0 & U|_{\mathcal{H}_{\text{nil}}} & 0 \\ 0 & 0 & U|_{\mathcal{H}_c} \end{bmatrix}$. ■

For convenience, and following Popescu [17], we introduce the following notation. Denote $\mathbb{Z}_+ \cup \{\infty\}$ by $\mathbb{N}_\infty$ and denote by $\mathcal{C}_n$ the set of all n-tuples of commuting row contractions on Hilbert spaces. We define $\varphi : \mathcal{C}_n \to \mathbb{N}_\infty \times \mathbb{N}_\infty \times \mathbb{N}_\infty$ as follows: Let T be an n-tuple of commuting row contraction on $\mathcal{H}$. Define

$$\varphi(T) = (p, m, q),$$

where $m := \deg \theta_T$, $q := \dim\{h \in \mathcal{H} : \sum_{|\alpha|=k} \|T^{*\alpha}h\|^2 = \|h\|^2, \text{ for all } k \in \mathbb{Z}_+\}$ and

$$p := \begin{cases} \dim(\mathcal{D}_m \ominus \mathcal{D}_{m+1}) & \text{if } m \in \mathbb{Z}_+ \\ \dim \mathcal{D}_{T^*} & \text{if } m = \infty, \end{cases}$$

and $\mathcal{D}_m := \overline{\text{span}}\{T^{\alpha}D_{T^*}h : h \in \mathcal{H}, |\alpha| \geq m\}$.

Clearly, if a pair T and $T^{'}$ in $\mathcal{C}_n$ are unitarily equivalent, then $\varphi(T) = \varphi(T^{'})$. For $T \in \mathcal{C}_n$ such that $\varphi(T) \in \mathbb{N}_\infty \times \{0\} \times \{0\}$, we have the following:

THEOREM 6.3. *Let $T, T' \in \mathcal{C}_n$.*

(i) *T is a Drury-Arveson shift if and only if T is regular and $\varphi(T) \in \mathbb{N}_\infty \times \{0\} \times \{0\}$.*
(ii) *If T and $T^{'}$ are regular and $\varphi(T) = \varphi(T^{'}) = (p, 0, 0)$ for some $p \in \mathbb{N}_\infty$, then T and $T^{'}$ are unitary equivalent and $rank D_{T^*} = rank D_{T^{'*}} = p$.*

Proof. (i) To prove the necessary part, without loss of generality, assume that T is M_z on $H_n^2(\mathcal{E})$ for some Hilbert space $\mathcal{E}$. Observe that since $\theta_{M_z} \equiv 0$, we have $m = 0$. Also note that $D_{T^*} = P_{\mathbb{C}} \otimes I_{\mathcal{E}}$, which implies $\mathcal{D}_{T^*} = \mathbb{C} \otimes \mathcal{E}$. Since

$$\overline{\text{span}}\{T^{\alpha}D_{T^*}h : h \in \mathcal{H}, \alpha \in \mathbb{Z}_+^n\} \ominus \overline{\text{span}}\{T^{\alpha}D_{T^*}h : h \in \mathcal{H}, |\alpha| \geq 1\} = \mathbb{C} \otimes \mathcal{E},$$

it follows that $p = \dim \mathcal{E} = \text{rank} D_{T^*} \in \mathbb{N}_\infty$. Since T is pure, $\mathcal{H}_c = \{0\}$, and so $q = \dim \mathcal{H}_c = 0$. Thus $\varphi(T) \in \mathbb{N}_\infty \times \{0\} \times \{0\}$. For the converse, assume that T is regular and $\varphi(T) \in \mathbb{N}_\infty \times \{0\} \times \{0\}$. Therefore, since $m = q = 0$, Theorem 3.6 implies that $\mathcal{H}_{\text{nil}} = \{0\}, \mathcal{H}_c = \{0\}$, that is, T is a Drury-Arveson shift.

(ii) This follows from part (i) and the fact that the multiplicity is a complete set of unitary invariant of Drury-Arveson shifts. ∎

The noncommutative version of the above result is due to Popescu [17, Theorem 2.2]. Now we turn to pure row contractions in $\mathcal{C}_n$. The proof is completely analogous to the proof of [17, Theorem 2.4 (i)].

Proposition 6.4. *Let T be an n-tuple of commuting row contraction with polynomial characteristic function. Then T is pure if and only if $\varphi(T) \in \mathbb{N}_\infty \times \mathbb{Z}_+ \times \{0\}$.*

Proof. Assume that T is pure. Consider the canonical representation of T on $\mathcal{M} \oplus \mathcal{H}_{\text{nil}} \oplus \mathcal{H}_c$ as in Theorem 3.6. For each $h \in \mathcal{H}_c$, it follows that $T^{*\alpha} h = W^{*\alpha} h$ and hence

$$\|h\|^2 = \sum_{|\alpha|=k} \|W^{*\alpha} h\|^2 = \sum_{|\alpha|=k} \|T^{*\alpha} h\|^2,$$

for all $k \in \mathbb{N}$. Since T is pure, this implies that $\mathcal{H}_c = \{0\}$, that is, $\varphi(T) \in \mathbb{N}_\infty \times \mathbb{Z}_+ \times \{0\}$. Conversely, if $\varphi(T) \in \mathbb{N}_\infty \times \mathbb{Z}_+ \times \{0\}$, then $\mathcal{H}_c = \{0\}$. The canonical representation of T as in Theorem 3.6 then becomes $T_i = \begin{bmatrix} M_i & * \\ 0 & N_i \end{bmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{H}_{\text{nil}}$. Suppose m is the order of the nilpotent operator N. Then for each $\alpha \in \mathbb{Z}_+^n$, $|\alpha| = m$, there exists $X_\alpha \in \mathcal{B}(\mathcal{H}_{\text{nil}}, \mathcal{M})$ such that $T^\alpha = \begin{bmatrix} M^\alpha & X_\alpha \\ 0 & 0 \end{bmatrix}$. By a computation similar to that in [17, Theorem 2.4 (i)], we obtain that T is pure. ∎

Along similar lines, most of Popescu's results in [17, Section 2] hold in a similar way for n-tuples of commuting contractions. We only point one which needs an additional assumption.

Theorem 6.5. *Let T be an n-tuple of commuting contractions on a Hilbert space with polynomial characteristic function. If T is regular, then the following are equivalent:*

(1) *θ_T is constant.*
(2) *$\varphi(T) \in \mathbb{N}_\infty \times \{0\} \times \mathbb{N}_\infty$.*
(3) *The canonical decomposition of T is given by: $T_i = \begin{bmatrix} M_i & * \\ 0 & W_i \end{bmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{H}_c$, $i = 1, \ldots, n$, where $(M_1, \ldots, M_n)$ is a Drury-Arveson shift on $\mathcal{M}$ and $(W_1, \ldots, W_n)$ is a spherical co-isometry on $\mathcal{H}_c$.*

Proof. The proof follows from the definition of the map φ and the canonical representation of the row contraction T with polynomial characteristic function. ∎

7. An example

In this section, we provide an example of a commuting tuple, which is a partial isometry with wandering subspace property, but whose characteristic function is not a polynomial.

Therefore, the tuple is not unitarily equivalent to a Drury Arveson shift. This justifies the presence of the regularity assumption in the Theorem 3.6.

We consider a subspace of H_2^2 which is invariant under the Drury-Arveson shift. That is,

$$\mathcal{M} := \bigoplus_{n\geq 2} \mathbb{H}_n \subseteq H_2^2,$$

where $\mathbb{H}_n$ denotes the class of homogeneous polynomials of degree n and a commuting pair of bounded linear operators $V = (V_1, V_2)$ on $\mathcal{M}$, defined by

$$V_i := M_{z_i}|_{\mathcal{M}} \qquad \text{for } i = 1, 2.$$

By using the definition of the adjoint of the Drury-Arveson shift, it is trivial to observe that for each $i = 1$ and 2,

$$V_i V_i^* \boldsymbol{z}^\alpha = \begin{cases} \frac{\alpha_i}{|\alpha|} \boldsymbol{z}^\alpha & \text{if } |\alpha| \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of V_i's, one can easily derive $D_{V^*}^2 = I - \sum_{i=1}^2 V_i V_i^* = P_{\mathbb{H}_2}$, where $P_{\mathbb{H}_2}$ is an orthogonal projection onto the subspace $\mathbb{H}_2$. Hence, V is a row contraction on $\mathcal{M}$, and we recall the expression of the characteristic function of V, given in 2.1, and the Taylor series expansion, that is,

$$\begin{aligned} \Theta_V(z) &= [-V + D_{V^*}(I - ZV^*)^{-1} Z D_V]|_{\mathcal{D}_V} \\ &= (-V + \sum_{|\alpha|\geq 1} \Theta_{V,\alpha} \boldsymbol{z}^\alpha)|_{\mathcal{D}_V}, \end{aligned}$$

where for each α with $|\alpha| \geq 1$ the coefficients $\Theta_{V,\alpha} = \sum_{i=1}^2 \gamma_{\alpha - e_i} D_{V^*} V^{*(\alpha - e_i)} P_i D_V$. Also due to the fact that, $\mathrm{Im} V \subseteq \bigoplus_{n\geq 3} \mathbb{H}_n$, we have $V D_V = D_{V^*} V = P_{\mathbb{H}_2} V = 0$.

On the other hand, from the definition of the defect operator $D_V^2 : \mathcal{M} \oplus \mathcal{M} \to \mathcal{M} \oplus \mathcal{M}$, the action on the elements $(z_1^{\alpha_1}, z_1^{\alpha_1})^{tr}$ with $\alpha_1 \geq 2$ is the following

$$D_V^2 \begin{bmatrix} z_1^{\alpha_1} \\ z_1^{\alpha_1} \end{bmatrix} = \begin{bmatrix} I - V_1^* V_1 & -V_1^* V_2 \\ -V_2^* V_1 & I - V_2^* V_2 \end{bmatrix} \begin{bmatrix} z_1^{\alpha_1} \\ z_1^{\alpha_1} \end{bmatrix} = \begin{bmatrix} \frac{-\alpha_1}{\alpha_1+1} z_1^{\alpha_1 - 1} z_2 \\ \frac{\alpha_1}{\alpha_1+1} z_1^{\alpha_1} \end{bmatrix}.$$

Now, for any $\alpha_1 \geq 2$, we consider $\beta = (\alpha_1 - 1, 0)$ and we have

$$\begin{aligned} \Theta_{V,\beta} \begin{bmatrix} \frac{-\alpha_1}{\alpha_1+1} z_1^{\alpha_1 - 1} z_2 \\ \frac{\alpha_1}{\alpha_1+1} z_1^{\alpha_1} \end{bmatrix} &= \gamma_{\beta - e_1} P_{\mathbb{H}_2} V_1^{*(\alpha_1 - 2)} \big(\frac{-\alpha_1}{\alpha_1 + 1} z_1^{\alpha_1 - 1} z_2 \big) \\ &= \gamma_{\beta - e_1} P_{\mathbb{H}_2} \big(- c_{\alpha_1} z_1 z_2 \big) \\ &= d_{\alpha_1} z_1 z_2, \end{aligned}$$

where $d_{\alpha_1} = -\gamma_{\beta - e_1} c_{\alpha_1}$ for some non-zero constant c_{α_1}. Hence, $\Theta_{V,\beta} \neq 0$. Moreover, we can conclude that for each $\alpha = (\alpha_1, 0)$ with $\alpha_1 \geq 2$, $\Theta_{V,\beta} \neq 0$ where $\beta = (\alpha_1 - 1, 0)$. In other words, there are infinitely many β's for which $\Theta_{V,\beta} \neq 0$, that is, the characteristic function Θ_V is not a polynomial.

Following the above calculation, it is straightforward to conclude that V is a pure partial isometry, but it is not unitary equivalent to Drury-Arveson shift as its characteristic function is not the zero polynomial. By [Corollary 3.10, [5]], it follows that the tuple $V = (V_1, V_2)$ is not regular in the sense of Definition 3.4.

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